THE PILA-WILKIE THEOREM AND ANALYTIC AX-KOCHEN-ERSOV THEORY

BY

NEER BHARDWAJ

University of Illinois at Urbana-Champaign

Abstract

In Part I, we consider the four structures $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$, and $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ where $\mathbb{Z}$ is the additive group of integers, $\mathrm{SF}^{\mathbb{Z}}$ is the set of $a \in \mathbb{Z}$ such that $v_p(a) < 2$ for every prime p and corresponding p-adic valuation v_p, $\mathbb{Q}$ and $\mathrm{SF}^{\mathbb{Q}}$ are defined likewise for rational numbers, and $<$ denotes the natural ordering on each of these domains. We prove that the second structure is model-theoretically wild while the other three structures are model-theoretically tame. Moreover, all these results can be seen as examples where number-theoretic randomness yields model-theoretic consequences.

Part II gives an account of the Pila-Wilkie counting theorem and some of its extensions and generalizations. We use semialgebraic cell decomposition to simplify part of the original proof. We also include complete treatments of a result due to Pila and Bombieri and of the o-minimal Yomdin-Gromov theorem that are used in this proof. For the latter we follow Binyamini and Novikov.

Part III develops an extension theory for *analytic* valuation rings in order to establish Ax-Kochen-Ersov type results for these structures. New is that we can add in salient cases lifts of the residue field and the value group and show that the induced structure on the lifted residue field is just its field structure, and on the lifted value group is just its ordered abelian group structure. This restores an analogy with the non-analytic AKE-setting that was missing in earlier treatments of analytic AKE-theory.

To Nana, for everything.

Acknowledgments

My deepest gratitude goes to my advisor, Lou van den Dries, for his guidance, care and support, and for his generosity and patience. Being his student and collaborator has been a truly edifying experience, one that I will cherish all my life.

I am also deeply indebted to Chieu-Minh Tran, my academic sibling, for his invaluable encouragement and deep support, and for serving really as a quasi-advisor towards the start of my doctoral program.

I would like to extend a very special thanks to Jeremiah Heller and Philipp Hieronymi, for being at the core of a productive and pleasant experience at UIUC. I am similarly grateful to Anush Tserunyan and Erik Walsberg for their enriching presence as part of the Logic group at UIUC, and to James Freitag for all his generous and helpful advice, and for serving on my doctoral committee.

I would also like to thank other students of Lou, Santiago Camacho, Allen Gehret, Tigran Hakobyan, Elliot Kaplan, and Nigel Pynn-Coates, for making my time at UIUC more engaging and enjoyable.

I was extremely privileged to have had the most special mentors during my formative years as an undergraduate. This thesis is partially dedicated to Arbind Lal, who left us far too soon. At IITK and elsewhere, Professors Mohua Banerjee, Sameer Chavan, Aparna Dar, Shobha Madan, Alok Maloo, Parasar Mohanty, Nandini Nilakantan, P Shunmugaraj, Maneesh Thakur, and Sandeep Varma all made absolutely invaluable and indelible contributions to my life and career, both inside and outside the classroom.

Last but far from the least, I want to present my heartfelt gratitude to Salik Ram Mishra, my high school math teacher, for his continuing contribution of belief and direction to my life.

Table of contents

CHAPTER 1

Introduction

This thesis comprises three parts corresponding to three projects. These projects all involve model theory to study issues around definability in various mathematical structures, but are otherwise unrelated. The first part is joint work with Chieu-Minh Tran and was published as [10]; *this part is not represented in the title of the thesis*. The second and third parts are joint work with Lou van den Dries. The former was published as [9] and the latter will be expanded to a paper to be submitted for publication.

This chapter gives a broad overview of my doctoral research, and each part of the thesis will contain a detailed introduction. The three sections below correspond to the parts of the thesis.

1.1 Groups $\mathbb{Z}$ and $\mathbb{Q}$ with square-freeness predicates

Throughout this thesis, $n \in \mathbb{N}$. In first-order model theory, we study the sets *definable* in a *structure*. In brief, a set is said to be definable in a structure, if it is given as an n-ary relation on the *domain* of the structure, and the relation is described by some *formula* in the first-order *language* of the structure. I will now give some examples for the jargon in italics, and leave it to the reader to arrive/look at precise formulations.

$(\mathbb{Z}; 0, 1, +, \cdot)$ is an example of a structure, its domain is $\mathbb{Z}$, and the language associated with the structure is $\{0, 1, +, \cdot\}$, the *language of rings*. For example, $\big(xy + 1 = 0\big)$ is a formula in the language, so is $\big(\forall y\ x^7y^2 + x^6y^4 + 1 = 0\big)$, and

$$\{x \in \mathbb{Z} :\ \text{for all } y,\ x^7y^2 + x^6y^4 + 1 = 0 \text{ and } xy + 1 \neq 0\}$$

is an example of a set definable in the structure $(\mathbb{Z}; 0, 1, +, \cdot)$.

So in particular with the structure $(\mathbb{Z}; 0, 1, +, \cdot)$, we can ask if certain diophantine equations have solutions or not. In view of Matiyasevich's famous resolution of Hilbert's tenth problem [46], the class of definable sets of $(\mathbb{Z}; 0, 1, +, \cdot)$ have a deep anarchic quality to them. The full class of sets definable in $(\mathbb{Z}; 0, 1, +, \cdot)$ is much broader than just sets of diophantine solutions of polynomials, and Gödel's incompleteness theorem is an undecidability result for this bigger class. For reasons not unrelated to the two blockbuster results above, the structure $(\mathbb{Z}; 0, 1, +, \cdot)$ is considered wild from other model-theoretic classification points of view as well.

With the integers being so obviously central to mathematics, structures with domain $\mathbb{Z}$, in particular expansions of the the group of integers, i.e. $(\mathbb{Z}; 0, 1, +)$, are an area of active and vigorous interest in model theory. Our main result in Part I is about the model-theoretic properties of $(\mathbb{Z}; 0, 1, +, \mathrm{SF}^{\mathbb{Z}})$, with $\mathrm{SF}^{\mathbb{Z}}$ being the set of square-free integers.

Theorem 1.1 (Theorem 2.1)**.** *The theory of* $(\mathbb{Z}; 0, 1, +, -, \mathrm{SF}^{\mathbb{Z}})$ *is model complete, decidable, unstable and supersimple of U-rank 1.*

Our result follows Kaplan and Shelah's work [41], where they expand the group of integers by a predicate for primes, and obtain the exact analogue of the above theorem for their structure; but under the assumption of Dickson's conjecture. We prove the analogue of Dickson's conjecture for the square-free integers required for our purposes; a purely analytic number theoretic fact. $(\mathbb{Z}; 0, 1, +, -, \mathrm{SF}^{\mathbb{Z}})$ is actually a rare example of a structure with domain $\mathbb{Z}$ whose theory is unconditionally known to be tame but not NIP.

We also consider $(\mathbb{Z}; 0, 1, +, -, <, \mathrm{SF}^{\mathbb{Z}})$, the expansion of the structure above with a binary relation for the usual ordering on the integers. The structure can define multiplication, and is hence *wild* by our discussion above on $(\mathbb{Z}; 0, 1, +, \cdot)$. This result is another analogue of a theorem for the corresponding structure with a predicate for primes [7]; this earlier result is again under the assumption of Dickson's conjecture.

Theorem 1.2 (Theorem 2.2)**.** *The theory of* $(\mathbb{Z}; 0, 1, +, -, <, \mathrm{SF}^{\mathbb{Z}})$ *interprets arithmetic, and hence is in particular undecidable.*

We also place two corresponding structures with domain the rational numbers on the tameness/decidability spectrum, see Theorems 2.3, 2.4.

1.2 O-minimality and the Pila-Wilkie Theorem

Going back to the language of rings, in contrast to the structure with domain $\mathbb{Z}$, the structure $(\mathbb{R}; 0, 1, +, \cdot)$ is model-theoretically tame, and its theory is also decidable; this is all based on classical work of Tarski [60]. Tarski's work, put into the relevant model-theoretic framework, also leads to an elegant reproof of Hilbert's 17^{th} problem, see [56]. The structure $(\mathbb{R}; 0, 1, +, \cdot)$, which is customarily referred to as the *real field*, is a principal example of what is called an *o-minimal* structure.

O-minimality as a subject within model theory started, in spirit, through van den Dries' work [25] on a problem of Tarski's from [60], was formally defined in [55], and has grown into a field of intense investigation and interest. The theory has had some spectacular applications to diophantine geometry and Hodge theory, which stem from the fact that the framework allows one to work with much larger classes of sets than in algebraic geometry, while keeping many of the familiar finiteness properties of real algebraic sets. Typically, such a class of sets contains all compact subanalytic sets, as well as the graph of the exponential function.

The Pila-Wilkie counting theorem [53] and variations have been cornerstones of strategies employed to attack several Zilber-Pink problems, a prime example being Pila's proof of the André-Oort conjecture for products of modular curves [52], which followed [54]. Moreover, an o-minimal Chow's theorem [49] was crucially involved in the proof Griffith's conjecture about images of period mappings [6].

In a nutshell, the Pila-Wilkie theorem gives a subpolynomial upper bound in terms of their heights, on the number of rational points inside the transcendental part of a set definable in an o-minimal expansion of the real field. Part II presents a complete exposition of the Pila-Wilkie theorem, and some of its extensions and generalizations. In fact, we exploit cell decomposition more thoroughly to simplify the deduction from the main ingredients of the original proof. The technically most demanding ingredient is an o-minimal Yomdin-Gromov theorem, and we include a simpler treatment of this result following Binyamini and Novikov [15], and add appendices on o-minimality and model theory to make our account self-contained.

The overarching principle with Pila-Wilkie type results is that *transcendental* sets contain *few* rational points. We now make these things precise so that we can state the theorem. For a measure on the density of rational points, we define the *multiplicative height function* $\mathrm{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathrm{H}(\frac{a}{b}) := \max(|a|, |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Let $n \geqslant 1$; then for $a = (a_1, \ldots, a_n) \in \mathbb{Q}^n$, we set

$$\mathrm{H}(a) := \max\{\mathrm{H}(a_i) : 1 \leqslant i \leqslant n\} \in \mathbb{N}^{\geqslant 1}.$$

For $X \subseteq \mathbb{R}^n$ set $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$, and for T ranging over real numbers $\geqslant 1$, $X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) : \mathrm{H}(a) \leqslant T\}$ is the (finite) set of rational points of X of height $\leqslant T$, and $N(X, T) := \#X(\mathbb{Q}, T) \in \mathbb{N}$.

We aim for sub-polynomial asymptotic upper bounds on $N(X, T)$, under certain geometric conditions on X. Such $X \subseteq \mathbb{R}^n$ may contain semialgebraic subsets of positive dimension even if X itself is not semialgebraic, and these semialgebraic subsets can contain cT^{δ} rational points of height at most T, for some c, $\delta > 0$ and all $T \geqslant 1$. To address this issue, we remove the *algebraic part* of X, denoted by X^{alg}, to be the union of the connected infinite semialgebraic subsets of X. Set $X^{\mathrm{tr}} := X \setminus X^{\mathrm{alg}}$; we can now state the Pila-Wilkie counting theorem:

Theorem 1.3 (Theorem 5.1)**.** *Let $X \subseteq \mathbb{R}^n$ be a set definable in an o-minimal expansion of the real field, and let $\varepsilon > 0$. There is a constant $C(X, \varepsilon) \in \mathbb{R}^{>0}$ such that*

$$N(X^{\mathrm{tr}}, T) \leqslant C(X, \varepsilon) T^{\varepsilon}.$$

As in the original, our proof too motivates more general family version(s), Theorems 5.7, 5.8. Moreover we apply our strategy to recover an extension, Theorem 8.4, and a couple of generalizations, Theorems 8.5, 8.9 which first appear in [51].

1.3 An analytic AKE program

The following result is an early famous example of using model theory in (p-adic) number theory, due to Ax and Kochen [3, 4, 5].

Theorem 1.4. *Given any natural number d, there is a natural number N such that for all prime numbers $p > N$, every homogeneous polynomial of degree d over the p-adic numbers in at least $d^2 + 1$ variables has a non-trivial zero.*

The key underlying result in [3, 4, 5] on henselian valued fields of equicharacteristic 0 was also independently proved by Ersov [33, 34, 35, 36] and gave rise to what we now call AKE-theory, extending and refining the original work in many ways. The AKE principle, stated loosely, is that the model theory of a henselian valued field is completely determined by the model theory of its residue field and value group.

In an extension due to Denef and van den Dries [23, 27] suitable henselian valued fields are equipped with extra first-order primitives given by restricted analytic functions, and this led to answering a question by Serre on p-adic analytic varieties. The key argument in [23, 27] is a direct reduction to the original AKE-setting by means of a piecewise uniform Weierstrass preparation result applied to the relevant restricted power series.

It has a drawback: The original AKE-proof by model theory and valuation theory allows one to show that for a lifting of the residue field and of the value group, (in the equicharacteristic 0 case) the induced structure on these lifts is just given by what is definable in the lift as a pure field, respectively, as a pure ordered abelian group. The direct reductions in [23, 27], however, do not give the corresponding *induced structure* theorem in the setting with restricted analytic functions.

In Part III we develop an extension theory for *analytic valuation rings*, in parallel with ordinary valuation theory, so that model theory is used to recover such an induced structure result. Weierstrass preparation is still crucial, but now serves only to develop the extension theory. We obtain an analytic AKE-type equivalence result, Theorem 12.14, and end this part of the thesis by using our technology to recover [12, Proposition 2] as an immediate consequence. The authors of [12] use it in connection with a counterexample to a possible Pila-Wilkie type result for definable sets of $\mathbb{C}((t))$ with restricted analytic functions.

Part I

The groups $\mathbb{Z}$ and $\mathbb{Q}$ with predicates for being square-free

CHAPTER 2

Motivations and the number theory

In this part of the thesis we explore the *randomness* of the square-free integers and evaluate whether four first-order structures are *tame* or not.

In Chapter 2 we state our results for the four structures, and isolate the technology around the *random* behaviour of the square-free integers of interest to us. We employ this number theoretic phenomenon to deduce model theoretic consequences for each of the structures in Chapter 3.

2.1 Our theorems

In [41], Kaplan and Shelah showed under the assumption of Dickson's conjecture that if $\mathbb{Z}$ is the additive group of integers implicitly assumed to contain the element 1 as a distinguished constant and the map $a \mapsto -a$ as a distinguished function, and if Pr is the set of $a \in \mathbb{Z}$ such that either a or $-a$ is prime, then the theory of $(\mathbb{Z}; \mathrm{Pr})$ is model complete, decidable, and super-simple of U-rank 1. From our current point of view, the above result can be seen as an example of a more general phenomenon where we can often capture aspects of randomness inside a structure using first-order logic and deduce in consequence several model-theoretic properties of that structure. In $(\mathbb{Z}; \mathrm{Pr})$, the conjectural randomness is that of the set of primes with respect to addition. Dickson's conjecture is useful here as it reflects this randomness in a fashion which can be made first-order. The second author's work in [61] provides another example with similar themes.

Our viewpoint in particular predicts that there are analogues of Kaplan and Shelah's results with Pr replaced by other random subsets of $\mathbb{Z}$. We confirm the above prediction in this part of the thesis without the assumption of any conjecture when Pr is replaced with the set

$$\mathrm{SF}^{\mathbb{Z}} = \{a \in \mathbb{Z} : \text{ for all } p \text{ primes, } v_p(a) < 2\}$$

where v_p is the p-adic valuation associated to the prime p. We have that $\mathbb{Z}$ is a structure in the language L of additive groups augmented by a constant symbol for 1 and a function symbol for $a \mapsto -a$. Then $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ is a structure in the language L_{u} extending L by a unary predicate symbol for $\mathrm{SF}^{\mathbb{Z}}$ (as indicated by the additional subscript "u"). We will introduce a first-order notion of *genericity* which captures the partial randomness in the interaction between $\mathrm{SF}^{\mathbb{Z}}$ and the additive structure on $\mathbb{Z}$. Using a similar idea as in [41], we obtain:

Theorem 2.1. *The theory of* $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ *is model complete, decidable, supersimple of U-rank* 1*, and is* k*-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

The above theorem gives us without assuming any conjecture the first natural example of a simple unstable expansion of $\mathbb{Z}$. From the same notion of *genericity*, we deduce entirely different consequences for the structure

$(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ in the language L_{ou} extending L_{u} by a binary predicate symbol for the natural ordering $<$ (as indicated by the additional subscript "o"):

Theorem 2.2. *The theory of* $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ *interprets arithmetic.*

The proof here is an adaption of the strategy used in [7] to show that the theory of $(\mathbb{N}; +, <, \mathrm{Pr})$ with Pr the set of primes interprets arithmetic. The above two theorems are in stark contrast with one another in view of the fact that $(\mathbb{Z}; <)$ is a minimal proper expansion of $\mathbb{Z}$; indeed, it is proven in [22] that adding any new definable set from $(\mathbb{Z}; <)$ to $\mathbb{Z}$ results in defining $<$. On the other hand, it is shown in [24] that there is no strong expansion of the theory of Presburger arithmetic, so Theorem 2.2 is perhaps not entirely unexpected.

It is also natural to consider the structures $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ where $\mathbb{Q}$ is the additive group of rational numbers, also implicitly assumed to contain 1 as a distinguished constant and $a \mapsto -a$ as a distinguished function,

$$\mathrm{SF}^{\mathbb{Q}} = \{a \in \mathbb{Q} : v_p(a) < 2 \text{ for all primes } p\},$$

and the relation $<$ on $\mathbb{Q}$ is the natural ordering. The reader might wonder why chose the above $\mathrm{SF}^{\mathbb{Q}}$ instead of $\mathrm{SF}^{\mathbb{Z}}$ or $\mathrm{ASF}^{\mathbb{Q}} = \{a \in \mathbb{Q} : |v_p(a)| < 2 \text{ for all primes } p\}$. From Lemma 2.6 in the next section, we get $\mathrm{SF}^{\mathbb{Q}} + \mathrm{SF}^{\mathbb{Q}} = \mathbb{Q}$, $\mathrm{SF}^{\mathbb{Z}} + \mathrm{SF}^{\mathbb{Z}} = \mathbb{Z}$, and $\mathrm{ASF}^{\mathbb{Q}} + \mathrm{ASF}^{\mathbb{Q}} = \{a : v_p(a) > -2 \text{ for all primes } p\}$. Hence, equipping $\mathbb{Q}$ and $(\mathbb{Q}; <)$ with either $\mathrm{SF}^{\mathbb{Z}}$ or $\mathrm{ASF}^{\mathbb{Q}}$ will result in structures expanding a infinite-index pair of infinite abelian groups with a unary predicate on the smaller group, and therefore, having rather different flavors from $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ and $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$.

Viewing $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ in the obvious way as an L_{u}-structure and an L_{ou}-structure, the main new technical aspect is in showing that these two structures satisfy suitable notions of *genericity* and leveraging on them to prove:

Theorem 2.3. *The theory of* $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ *is model complete, decidable, simple but not supersimple, and is* k*-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

From above, $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ is "less tame" than $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$. The reader might therefore expect that $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ is wild. However, this is not the case:

Theorem 2.4. *The theory* $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ *is model complete, decidable, is* NTP_2 *but is not strong, and is* k*-independent for all* $k \in \mathbb{N}^{\geqslant 1}$.

Notation and conventions. Let h, k and l range over the set of integers and let m, n, and n' range over the set of natural numbers (which include zero). We let p range over the set of prime numbers, and denote by v_p the p-adic valuation on $\mathbb{Q}$. Let x be a single variable, y a tuple of variables of unspecified length, z the tuple $(z_1, \ldots, z_n)$ of variables, and z' the tuple $(z'_1, \ldots, z'_{n'})$ of variables. For an n-tuple a of elements from a certain set, we let a_i denote the i-th component of a for $i \in \{1, \ldots, n\}$. Suppose G is an additive abelian group. We equip G^m with a group structure by setting $+$ on G^m to be the coordinate-wise addition. Viewing G and G^m as $\mathbb{Z}$-module, we define ka with $a \in G$ and kb with $b \in G^m$ accordingly. Suppose, G is moreover an L-structure with 1_G the distinguished constant. We write k for $k1_G$. For $A \subseteq G$, we let $L(A)$ denote the language extending L by adding constant symbols for elements of A and view G as an $L(A)$ structure in the obvious way.

2.2 Genericity of the examples

In this section, we define the appropriate notions of *genericity* for the structures under consideration.

We study the structure $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ indirectly by looking at its definable expansion to a richer language. For given p and l, set

$$U_{p,l}^{\mathbb{Z}} = \{a \in \mathbb{Z} : v_p(a) \geqslant l\}.$$

Let $\mathscr{U}^{\mathbb{Z}} = (U_{p,l}^{\mathbb{Z}})$. The definition for $l \leqslant 0$ is not too useful as $U_{p,l}^{\mathbb{Z}} = \mathbb{Z}$ in this case. However, we still keep this for the sake of uniformity as we treat $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ later. For $m > 0$, set

$$P_m^{\mathbb{Z}} = \{a \in \mathbb{Z} : v_p(a) < 2 + v_p(m) \text{ for all } p\}.$$

In particular, $P_1^{\mathbb{Z}} = \mathrm{SF}^{\mathbb{Z}}$. Let $\mathscr{P}^{\mathbb{Z}} = (P_m^{\mathbb{Z}})_{m>0}$. We have that $(\mathbb{Z}, \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ is a structure in the language L_{u}^* extending L_{u} by families of unary predicate symbols for $\mathscr{U}^{\mathbb{Z}}$ and $(P_m^{\mathbb{Z}})_{m>1}$. Note that

$$U_{p,l}^{\mathbb{Z}} = \mathbb{Z} \text{ for } l \leqslant 0, \quad U_{p,l}^{\mathbb{Z}} = p^l\mathbb{Z} \text{ for } l > 0, \text{ and } \quad P_m^{\mathbb{Z}} = \bigcup_{d|m} d\mathrm{SF}^{\mathbb{Z}} \text{ for } m > 0.$$

Hence, $U_{p,l}^{\mathbb{Z}}$ and $P_m^{\mathbb{Z}}$ are definable in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$, and so a subset of $\mathbb{Z}$ is definable in $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ if and only if it is definable in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$.

Let $(G; \mathscr{P}^G, \mathscr{U}^G)$ be an L_{u}^*-structure. Then $\mathscr{U}^G$ is a family indexed by pairs (p, l), and $\mathscr{P}^G$ is a family indexed by m. For p, l, and m, define $U_{p,l}^G \subseteq G$ to be the member of $\mathscr{U}^G$ with index (p, l) and $P_m^G \subseteq G$ to be the member of the family $\mathscr{P}^G$ with index m. In particular, we have $\mathscr{U}^G = (U_{p,l}^G)$ and $P^G = (P_m^G)_{m>0}$. Clearly, this generalizes the previous definition for $\mathbb{Z}$.

We isolate the basic first-order properties of $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. Let $\mathrm{Sf}_{\mathbb{Z}}^*$ be a recursive set of L_{u}^*-sentences such that an L_{u}^*-structure $(G; \mathscr{U}^G, \mathscr{P}^G)$ is a model of $\mathrm{Sf}_{\mathbb{Z}}^*$ if and only if $(G; \mathscr{U}^G, \mathscr{P}^G)$ satisfies the following properties:

(Z1) $(G; +, -, 0, 1)$ is elementarily equivalent to $(\mathbb{Z}; +, -, 0, 1)$;

(Z2) $U_{p,l}^G = G$ for $l \leqslant 0$, and $U_{p,l}^G = p^l G$ for $l > 0$;

(Z3) 1 is in P_1^G;

(Z4) for any given p, we have that $pa \in P_1^G$ if and only if $a \in P_1^G$ and $a \notin U_{p,1}^G$;

(Z5) $P_m^G = \bigcup_{d|m} dP_1^G$ for all $m > 0$.

The fact that we could choose $\mathrm{Sf}_{\mathbb{Z}}^*$ to be recursive follows from the well-known decidability of $\mathbb{Z}$. Clearly, $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ is a model of $\mathrm{Sf}_{\mathbb{Z}}^*$. Several properties which hold in $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ also hold in an arbitrary model of $\mathrm{Sf}_{\mathbb{Z}}^*$:

Lemma 2.5. *Let $(G; \mathscr{U}^G, \mathscr{P}^G)$ be a model of $\mathrm{Sf}_{\mathbb{Z}}^*$. Then we have the following:*

(i) *$(G; \mathscr{U}^G)$ is elementarily equivalent to $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}})$;*

(ii) *for all k, p, l, and $m > 0$, we have that*

$$k \in U_{p,l}^G \text{ if and only if } k \in U_{p,l}^{\mathbb{Z}} \quad \text{and} \quad k \in P_m^G \text{ if and only if } k \in P_m^{\mathbb{Z}};$$

(iii) *for all* $h \neq 0$, p, *and* l, *we have that* $ha \in U^G_{p,l}$ *if and only if* $a \in U^G_{p,l-v_p(h)}$;

(iv) *if* $a \in G$ *is in* $U^G_{p,2+v_p(m)}$ *for some* p, *then* $a \notin P^G_m$;

(v) *for all* $h \neq 0$ *and* $m > 0$, $ha \in P^G_m$ *if and only if we have*

$$a \in P^G_m \quad \textit{and} \quad a \notin U^G_{p,2+v_p(m)-v_p(h)} \textit{ for all } p \textit{ which divides } h;$$

(vi) *for all* $h > 0$ *and* $m > 0$, $a \in P^G_m$ *if and only if* $ha \in P^G_{mh}$.

Proof. Fix a model $(G; \mathscr{U}^G, \mathscr{P}^G)$ of $\mathrm{Sf}^*_{\mathbb{Z}}$. It follows from (Z2) that the same first-order formula defines both $U^G_{p,l}$ in G and $U^{\mathbb{Z}}_{p,l}$ in $\mathbb{Z}$. Then using (Z1), we get (i). The first assertion of (ii) is immediate from (i). Using this, (Z3), and (Z4), we get the second assertion of (ii) for the case $m = 1$. For $m \neq 1$, we reduce to the case $m = 1$ using property (Z5). Statement (iii) is an immediate consequence of (i). We only prove below the cases $m = 1$ of (iv $-$ vi) as the remaining cases of the corresponding statements can be reduced to these using (Z5). Statement (iv) is immediate for the case $m = 1$ using (Z2) and (Z4). The case $m = 1$ of (v) is precisely the statement of (Z4) when h is prime, and then the proof proceeds by induction. For the case $m = 1$ of (vi), ($\rightarrow$) follows from (Z5), and ($\leftarrow$) follows through a combination of Z5, (v) and induction on the number of prime divisors of h. $\square$

We next consider the structures $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$. For given p, l, and $m > 0$, in the same fashion as above, we set

$$U^{\mathbb{Q}}_{p,l} = \{a \in \mathbb{Q} : v_p(a) \geqslant l\} \quad \text{and} \quad P^{\mathbb{Q}}_m = \{a \in \mathbb{Q} : v_p(a) < 2 + v_p(m) \text{ for all } p\},$$

and let

$$\mathscr{U}^{\mathbb{Q}} = (U^{\mathbb{Q}}_{p,l}) \quad \text{and} \quad \mathscr{P}^{\mathbb{Q}} = (P^{\mathbb{Q}}_m)_{m>0}.$$

Then $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ is a structure in the language L^*_{u}. Clearly, every subset of $\mathbb{Q}^n$ definable in $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ is also definable in $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$. A similar statement holds for $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$. We will show that the reverse implications are also true.

The next lemma backs up the discussion on $\mathrm{SF}^{\mathbb{Q}}$ and $\mathrm{ASF}^{\mathbb{Q}}$ preceding Theorem 2.3 in the introduction.

Lemma 2.6. $\mathrm{SF}^{\mathbb{Z}} + \mathrm{SF}^{\mathbb{Z}} = \mathbb{Z}$, $\mathrm{SF}^{\mathbb{Q}} + \mathrm{SF}^{\mathbb{Q}} = \mathbb{Q}$, *and* $\mathrm{ASF}^{\mathbb{Q}} + \mathrm{ASF}^{\mathbb{Q}} = \{a : v_p(a) > -2 \textit{ for all } p\}$.

Proof. We first prove that any integer k is a sum of two elements from $\mathrm{SF}^{\mathbb{Z}}$. As $\mathrm{SF}^{\mathbb{Z}} = -\mathrm{SF}^{\mathbb{Z}}$ and the cases where $k = 0$ or $k = 1$ are immediate, we assume that $k > 1$. It follows from [57] that the number of square-free positive integers less than k is at least $\frac{53k}{88}$. Since $\frac{53}{88} > \frac{1}{2}$, this implies k can be written as a sum of two positive square-free integers which gives us $\mathrm{SF}^{\mathbb{Z}} + \mathrm{SF}^{\mathbb{Z}} = \mathbb{Z}$. Using this, the other two equalities follow immediately. $\square$

Lemma 2.7. *For all* p *and* l, $U^{\mathbb{Q}}_{p,l}$ *is existentially* 0*-definable in* $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$.

Proof. As $U^{\mathbb{Q}}_{p,l+n} = p^n U^{\mathbb{Q}}_{p,l}$ for all l and n, it suffices to show the statement for $l = 0$. Fix a prime p. We have for all $a \in \mathrm{SF}^{\mathbb{Q}}$ that

$$v_p(a) \geqslant 0 \ \text{ if and only if } p^2 a \notin \mathrm{SF}^{\mathbb{Q}}.$$

Using Lemma 2.6, for all $a \in \mathbb{Q}$, we have that $v_p(a) \geqslant 0$ if and only if there are $a_1, a_2 \in \mathbb{Q}$ such that

$$\big(a_1 \in \mathrm{SF}^{\mathbb{Q}} \wedge v_p(a_1) \geqslant 0\big) \wedge \big(a_2 \in \mathrm{SF}^{\mathbb{Q}} \wedge v_p(a_2) \geqslant 0\big) \text{ and } a = a_1 + a_2.$$

Hence, the set $U^{\mathbb{Q}}_{p,0} = \{a \in \mathbb{Q} : v_p(a) \geqslant 0\}$ is existentially definable in $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$. The desired conclusion follows. □

It is also easy to see that for all m, $P^{\mathbb{Q}}_m = m\mathrm{SF}^{\mathbb{Q}}$ for all $m > 0$, and so $P^{\mathbb{Q}}_m$ is existentially 0-definable in $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$. Combining with Lemma 2.7, we get:

Proposition 2.8. *Every subset of $\mathbb{Q}^n$ definable in $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ is also definable in $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$. The corresponding statement for $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ holds.*

In view of the first part of Proposition 2.8, we can analyze $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ via $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ in the same way we analyze $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ via $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. Let $\mathrm{Sf}^*_{\mathbb{Q}}$ be a recursive set of L^*_{u}-sentences such that an L^*_{u}-structure $(G; \mathscr{U}^G, \mathscr{P}^G)$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$ if and only if $(G; \mathscr{U}^G, \mathscr{P}^G)$ satisfies the following properties:

(Q1) $(G; +, -, 0, 1)$ is elementarily equivalent to $(\mathbb{Q}; +, -, 0, 1)$;

(Q2) for any given p, $U^G_{p,0}$ is an n-divisible subgroup of G for all n coprime with p;

(Q3) $1 \in U^G_{p,0}$ and $1 \notin U^G_{p,1}$;

(Q4) for any given p, $p^{-l}U^G_{p,l} = U^G_{p,0}$ if $l < 0$ and $U_{p,l} = p^l U_{p,0}$ if $l > 0$;

(Q5) $U^G_{p,0}/U^G_{p,1}$ is isomorphic as a group to $\mathbb{Z}/p\mathbb{Z}$;

(Q6) $1 \in P^G_1$;

(Q7) for any given p, we have that $pa \in P^G_1$ if and only if $a \in P^G_1$ and $a \notin U^G_{p,1}$;

(Q8) $P^G_m = mP^G_1$ for $m > 0$;

The fact that we could choose $\mathrm{Sf}^*_{\mathbb{Q}}$ to be recursive follows from the well-known decidability of $\mathbb{Q}$. Obviously, $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$. Several properties which hold in $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ also hold in an arbitrary model of $\mathrm{Sf}^*_{\mathbb{Q}}$:

Lemma 2.9. *Let $(G; \mathscr{U}^G, \mathscr{P}^G)$ be a model of $\mathrm{Sf}^*_{\mathbb{Q}}$. Then we have the following:*

(i) *For all p and all $l, l' \in \mathbb{Z}$ with $l \leqslant l'$, we have $U^G_{p,l}$ is a subgroup of G, $U^G_{p,l'} \subseteq U^G_{p,l}$. Further, we can interpret $U^G_{p,l}/U^G_{p,l'}$ as an L-structure with 1 being $p^l + U^G_{p,l'}$, and*

$$U^G_{p,l}/U^G_{p,l'} \cong_L \mathbb{Z}/(p^{l'-l}\mathbb{Z});$$

(ii) *for all h, $k \neq 0$, p, l, and $m > 0$, we have that*

$$\frac{h}{k} \in U^G_{p,l} \text{ if and only if } \frac{h}{k} \in U^{\mathbb{Q}}_{p,l} \quad \text{and} \quad \frac{h}{k} \in P^G_m \text{ if and only if } \frac{h}{k} \in P^{\mathbb{Q}}_m$$

where hk^{-1} is the obvious element in $\mathbb{Q}$ and in G;

(iii) *the replica of* (iii – vi) *of Lemma 2.5 holds.*

Proof. Fix a model $(G;\mathscr{U}^G,\mathscr{P}^G)$ of $\mathrm{Sf}^*_{\mathbb{Q}}$. From (Q2) we have that $U^G_{p,0}$ is a subgroup of G for all p. It follows from (Q4) that $U^G_{p,l'}\subseteq U^G_{p,l}$ are subgroups of G for all p and $l\leqslant l'$. With $U^G_{p,l}/U^G_{p,l'}$ being interpreted as an L-structure with 1 being $p^l+U^G_{p,l'}$, we get an L-embedding of $\mathbb{Z}/(p^{l'-l}\mathbb{Z})$ into $U^G_{p,l}/U^G_{p,l'}$ using (Q3) and (Q4). Further, we see that $|U^G_{p,l}/U^G_{p,l'}|=p^{(l'-l)}$ using (Q2)-(Q5) and induction on $l'-l$ together with the third isomorphism theorem; and so the aforementioned embedding must be an isomorphism, finishing the proof for (i). The first assertion of (ii) follows easily from (Q2)-Q(4). The second assertion for the case $m=1$ follows from the first assertion, (Q6), and (Q7). Finally, the case with $m\neq 1$ follows from the case $m=1$ using (Q8). The proof for the replica of (iii) from Lemma 2.5 is a consequence of (i) and (Q4). The proofs for replicas of (iv − vi) from Lemma 2.5 are similar to the proofs for (iv − vi) of Lemma 2.5. □

As the reader may expect by now, we will study $(\mathbb{Q};<,\mathrm{SF}^{\mathbb{Q}})$ via $(\mathbb{Q};<,\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$. Let L^*_{ou} be $L_{\mathrm{ou}}\cup L^*_{\mathrm{u}}$. Then $(\mathbb{Q};<,\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$ can be construed as an L^*_{ou}-structure in the obvious way. Let $\mathrm{OSf}^*_{\mathbb{Q}}$ be a recursive set of L^*_{ou}-sentences such that an L^*_{ou}-structure $(G;\mathscr{U}^G,\mathscr{P}^G)$ is a model of $\mathrm{OSf}^*_{\mathbb{Q}}$ if and only if $(G;\mathscr{U}^G,\mathscr{P}^G)$ satisfies the following properties:

1. $(G;<)$ is elementarily equivalent to $(\mathbb{Q};<)$;
2. $(G;\mathscr{U}^G,\mathscr{P}^G)$ is a model of $\mathrm{Sf}^*_{\mathbb{Q}}$.

As $\mathrm{Th}(\mathbb{Q};<)$ is decidable, we could choose $\mathrm{OSf}^*_{\mathbb{Q}}$ to be recursive.

Returning to the theory $\mathrm{Sf}^*_{\mathbb{Z}}$, we see that it does not fully capture all the first-order properties of $(\mathbb{Z},\mathscr{U}^{\mathbb{Z}},\mathscr{P}^{\mathbb{Z}})$. For instance, we will show later in Corollary 2.16 that for all $c\in\mathbb{Z}$, there is $a\in\mathbb{Z}$ such that

$$a+c\in\mathrm{SF}^{\mathbb{Z}}\ \text{ and }\ a+c+1\in\mathrm{SF}^{\mathbb{Z}},$$

while the interested reader can construct models of $\mathrm{Sf}^*_{\mathbb{Z}}$ where the corresponding statement is not true. Likewise, the theories $\mathrm{Sf}^*_{\mathbb{Q}}$ and $\mathrm{OSf}^*_{\mathbb{Q}}$ do not fully capture all the first-order properties of $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$ and $(\mathbb{Q};<,\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$.

To give a precise formulation of the missing first-order properties of $(\mathbb{Z},\mathscr{U}^{\mathbb{Z}},\mathscr{P}^{\mathbb{Z}})$, $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$, and $(\mathbb{Q};<\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$, we need more terminologies. Let $t(z)$ be an L^*_{u}-term (or equivalently an L^*_{ou}-term) with variables in z. An L^*_{u}-formula (or an L^*_{ou}-formula) which is a boolean combination of formulas having the form $t(z)=0$ where we allow t to vary is called an **equational condition**. Similarly, an L^*_{ou}-formula which is a boolean combination of formulas having the form $t(z)<0$ where t is allowed to vary is called an **order-condition**. For any given p, l define $t(z)\in U_{p,l}$ to be the obvious formula in $L^*_{\mathrm{u}}(z)$ which defines in an arbitrary L^*_{u}-structure $(G;\mathscr{U}^G,\mathscr{P}^G)$ the set

$$\{c\in G^n : t^G(c)\in U^G_{p,l}\}.$$

Define the quantifier-free formulas $t(z)\notin U_{p,l}$, $t(z)\in P_m$, and $t(z)\notin P_m$ in $L^*_{\mathrm{u}}(z)$ for p, l, and for $m>0$ likewise. For each prime p, an L^*_{u}-formula (or an L^*_{ou}-formula) which is a boolean combination of formulas of the form $t(z)\notin U_{p,l}$ where t and l are allowed to vary is called a ***p*-condition**. We call a p-condition as in the previous statement **trivial** if the boolean combination is the empty conjunction.

A **parameter choice** of variable type (x,z,z') is a triple (k,m,Θ) such that k is in $\mathbb{Z}\setminus\{0\}$, m is in $\mathbb{N}^{\geqslant 1}$, and $\Theta=\big(\theta_p(x,z,z')\big)$ where $\theta_p(x,z,z')$ is a p-condition for each prime p and is trivial for all but finitely many p. We say that an L^*_{u}-formula $\psi(x,z,z')$ is **special** if it has the form

$$\bigwedge_p \theta_p(x,z,z') \wedge \bigwedge_{i=1}^{n}(kx+z_i \in P_m) \wedge \bigwedge_{i'=1}^{n'}(kx+z'_i \notin P_m)$$

where k, m and $\theta_p(x,z,z')$ are taken from a parameter choice of variable type (x,z,z'). Every special formula corresponds to a unique parameter choice and vice versa. Special formulas are special enough that we have a "local to global" phenomenon in the structures of interest but general enough to represent quantifier free formulas. We will explain the former point in the remaining part of the section and make the latter point precise with Theorem 3.1.

Let $\psi(x,z,z')$ be a special formula with parameter choice (k,m,Θ) and $\theta_p(x,z,z')$ is the p-condition in Θ for each p. We define the **associated equational condition** of $\varphi(x,z,z')$ to be the formula

$$\bigwedge_{i=1}^{n}\bigwedge_{i'=1}^{n'}(z_i \neq z'_{i'})$$

and the **associated p-condition** of $\varphi(x,z,z')$ to be the formula

$$\theta_p(x,z,z') \wedge \bigwedge_{i=1}^{n}(kx+z_i \notin U_{p,2+v_p(m)}).$$

It is easy to see that modulo $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, an arbitrary special formula implies its associated equational condition and its associated p-condition for any prime p.

Suppose $(G;\mathscr{U}^G,\mathscr{P}^G)$ and $(H;\mathscr{U}^H,\mathscr{P}^H)$ are L^*_{u}-structures such that the former is an L^*_{u}-substructure of the latter. Let $\psi(x,z,z')$ be a special formula, $\psi_=(z,z')$ the associated equational condition, and $\psi_p(x,z,z')$ the associated p-condition for any given prime p. For $c \in G^n$ and $c' \in G^{n'}$, we call the quantifier-free $L^*_{\mathrm{u}}(G)$-formula $\psi(x,c,c')$ a **G-system**. An element $a \in H$ such that $\psi(a,c,c')$ holds is called a **solution** of $\psi(x,c,c')$ in H. We say that $\psi(x,c,c')$ is **satisfiable** in H if it has a solution in H and **infinitely satisfiable** in H if it has infinitely many solutions in H. We say that $\psi(x,c,c')$ is **nontrivial** if $\psi_=(c,c')$ holds or more explicitly if c and c' have no common components. For a given p, we say that $\psi(x,c,c')$ is **p-satisfiable** in H if there is $a_p \in H$ such that $\psi_p(a_p,c,c')$ holds. A G-system is **locally satisfiable** in H if it is p-satisfiable in H for all p.

Suppose $(G;<,\mathscr{U}^G,\mathscr{P}^G)$ and $(H;<,\mathscr{U}^H,\mathscr{P}^H)$ are L^*_{ou}-structures such that the former is an L^*_{ou}-substructure of the latter. All the definitions in the previous paragraph have obvious adaptations to this new setting as $(G;\mathscr{U}^G,\mathscr{P}^G)$ and $(H;\mathscr{U}^H,\mathscr{P}^H)$ are L^*_{u}-structures. For b and b' in H such that $b<b'$, define

$$(b,b')^H = \{a \in H : b < a < b'\}.$$

A G-system $\psi(x,c,c')$ is **satisfiable in every H-interval** if it has a solution in the interval $(b,b')^H$ for all b and b' in H such that $b<b'$. The following observation is immediate:

Lemma 2.10. *Suppose $(G;\mathscr{U}^G,\mathscr{P}^G)$ is a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$. Then every G-system which is satisfiable in G is nontrivial and locally satisfiable in G.*

It turns out that the converse and more are also true for the structures of interest. We say that a model

$(G; \mathscr{U}^G, \mathscr{P}^G)$ of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$ is **generic** if every nontrivial locally satisfiable G-system is infinitely satisfiable in G. An $\mathrm{OSf}^*_{\mathbb{Q}}$ model $(G; <, \mathscr{U}^G, \mathscr{P}^G)$ is **generic** if every nontrivial locally satisfiable G-system is satisfiable in every G-interval. We will later show that $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$, $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$, and $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ are generic.

Before that we will show that the above notions of *genericity* are first-order. Let $\psi(x, z, z')$ be the special formula corresponding to a parameter choice (k, m, Θ) with $\Theta = \big(\theta_p(x, z, z')\big)$. A **boundary** of $\psi(x, z, z')$ is a number $B \in \mathbb{N}^{>0}$ such that $B > \max\{|k|, n\}$ and $\theta_p(x, z, z')$ is trivial for all $p > B$.

Lemma 2.11. *Let $\psi(x, z, z')$ be a special formula, B a boundary of $\psi(x, z, z')$, and $(G; \mathscr{U}^G, \mathscr{P}^G)$ a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$. Then every G-system $\psi(x, c, c')$ is p-satisfiable for $p > B$.*

Proof. Let $\psi(x, z, z')$ be the special formula corresponding to a parameter choice (k, m, Θ), and B, $(G; \mathscr{U}^G, \mathscr{P}^G)$ as in the statement of the lemma. Suppose $\psi(x, c, c')$ is a G-system, $p > B$, and $\psi_p(x, z, z')$ is the associated p-condition of $\psi(x, z, z')$. Then $\psi_p(x, c, c')$ is equivalent to

$$\bigwedge_{i=1}^{n} (kx + c_i \notin U_{p, 2+v_p(m)}) \text{ in } (G; \mathscr{U}^G, \mathscr{P}^G).$$

We will show a stronger statement that there is a $a_p \in \mathbb{Z}$ satisfying the latter. Note that for all $d \notin U^G_{p,0}$, we have that $(ka + d \notin U_{p,0})$ for all $a \in \mathbb{Z}$. From Lemma 2.9, we have that $U^G_{p,l} \subseteq U^G_{p,k}$ whenever $k < l$, so we can assume that $c_i \in U^G_{p,0}$ for $i \in \{1, \dots, n\}$. In light of Lemma 2.5 (i) and Lemma 2.9 (i), we have that

$$U^G_{p,0} / U^G_{p, 2+v_p(m)} \cong_L \mathbb{Z}/(p^{2+v_p(m)}\mathbb{Z}).$$

It is easy to see that k is invertible mod $p^{2+v_p(m)}$ and that $p^{2+v_p(m)} > n$. Choose a_p in $\{0, \dots, p^{2+v_p(m)} - 1\}$ such that the images of $ka_p + c_1, \dots, ka_p + c_n$ in $\mathbb{Z}/(p^{2+v_p(m)}\mathbb{Z})$ are not 0. We check that a_p is as desired. ☐

Corollary 2.12. *There is an L^*_{u}-theory $\mathrm{SF}^*_{\mathbb{Z}}$ such that the models of $\mathrm{SF}^*_{\mathbb{Z}}$ are the generic models of $\mathrm{Sf}^*_{\mathbb{Z}}$. Similarly, there is an L^*_{u}-theory $\mathrm{SF}^*_{\mathbb{Q}}$ and an L^*_{ou}-theory $\mathrm{OSF}^*_{\mathbb{Q}}$ satisfying the corresponding condition for $\mathrm{Sf}^*_{\mathbb{Q}}$ and $\mathrm{OSf}^*_{\mathbb{Q}}$.*

In the rest of this part of the thesis, we fix $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ to be as in the previous lemma. We can moreover arrange them to be recursive. In the remaining part of this section, we will show that $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$, $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ and $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Z}})$ are models of $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ respectively. The proof that the latter are in fact the full axiomatizations of the theories of the former needs to wait until Section 3.1. Further we fix $\mathrm{SF}_{\mathbb{Z}}$ and $\mathrm{SF}_{\mathbb{Q}}$ to be the theories whose models are precisely the L_{u}-reducts of models of $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ respectively, and $\mathrm{OSF}_{\mathbb{Q}}$ to be the theory whose models are precisely L_{ou} reducts of models of $\mathrm{OSF}^*_{\mathbb{Q}}$. For the reader's reference, the following table lists all the languages, the corresponding theories and primary structures under consideration:

Languages	Theories	Primary structures
L	$\mathrm{Th}(\mathbb{Z})$, $\mathrm{Th}(\mathbb{Q})$	$\mathbb{Z}, \mathbb{Q}$
L_{u}	$\mathrm{SF}_{\mathbb{Z}}$, $\mathrm{SF}_{\mathbb{Q}}$	$(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$
L_{ou}	$\mathrm{OSF}_{\mathbb{Q}}$	$(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$, $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$
L^*_{u}	$\mathrm{Sf}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{Sf}^*_{\mathbb{Q}}$, $\mathrm{SF}^*_{\mathbb{Q}}$	$(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$, $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$
L^*_{ou}	$\mathrm{OSf}^*_{\mathbb{Q}}$, $\mathrm{OSF}^*_{\mathbb{Q}}$	$(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$

Suppose $h \neq 0$. For a term $t(z) = k_1 z_1 + \ldots + k_n z_n + e$, let $t^h(z)$ be the term $k_1 z_1 + \ldots + k_n z_n + he$. If $\varphi(z)$ is a boolean combination of atomic formulas of the form $t(z) \in U_{p,l}$ or $t(z) \in P_m$ where $t(z)$ is an L^*_{u}-term, define $\varphi^h(z)$ to be the formula obtained by replacing $t(z) \in U_{p,l}$ and $t(z) \in P_m$ in $\varphi(z)$ with $t^h(z) \in U_{p,l+v_p(h)}$ and $t^h(z) \in P_{mh}$ for every choice of p, l, m and L^*_{u}-term t. It follows from Lemma 2.5 (iii), (vi) and Lemma 2.9 (iii) that across models of $\mathrm{Sf}^*_{\mathbb{Z}}$ and $\mathrm{Sf}^*_{\mathbb{Q}}$,

$$\varphi^h(hz) \text{ is equivalent to } \varphi(z).$$

Moreover, if $\theta(z)$ is a p-condition, then $\theta^h(z)$ is also p-condition. If $\psi(x, z, z')$ is the special formula corresponding to a parameter choice (k, m, Θ) with $\Theta = \big(\theta_p(x, z, z')\big)$, then $\psi^h(x, z, z')$ is the special formula corresponding to the parameter choice (k, hm, Θ^h) with $\Theta^h = \big(\theta^h_p(x, z, z')\big)$. It is easy to see from here that:

Lemma 2.13. *For $h \neq 0$, any boundary of a special formula $\psi(x, z, z')$ is also a boundary of $\psi^h(x, z, z')$ and vice versa.*

Let $\psi(x, z, z')$ be a special formula, $(G; \mathscr{U}^G, \mathscr{P}^G)$ a model of either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, and $\psi(x, c, c')$ a G-system. Then $\psi^h(x, hc, hc')$ is also a G-system which we refer to as the h-**conjugate** of $\psi(x, c, c')$. This has the property that $\psi^h(ha, hc, hc')$ if and only if $\psi(a, c, c')$ for all $a \in G$.

For a and b in $\mathbb{Z}$, we write $a \equiv_n b$ if a and b have the same remainder when divided by n. We need the following version of Chinese remainder theorem:

Lemma 2.14. *Suppose B is in $\mathbb{N}^{>0}$, Θ is a family $\big(\theta_p(x, z)\big)_{p \leqslant B}$ of L^*_{u}-formulas with $\theta_p(x, z)$ being a p-condition for each $p \leqslant B$, and $c \in \mathbb{Z}^n$ is such that $\theta_p(x, c)$ defines a nonempty set in $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ for all $p \leqslant B$. Then we can find $D \in \mathbb{N}^{>0}$ and $r \in \{0, \ldots, D-1\}$ such that for all $h \neq 0$ with $\gcd(h, B!) = 1$, we have*

$$a \equiv_D hr \text{ implies } \bigwedge_{p \leqslant B} \theta^h_p(a, hc) \quad \text{for all } a \in \mathbb{Z}.$$

Proof. Let B, Θ, and c be as stated. Fix $h \neq 0$ such that $\gcd(h, B!) = 1$. Hence, $v_p(h) = 0$ for $p \leqslant B$, and so the p-condition $\theta^h_p(x, z)$ is obtained from the p-condition $\theta_p(x, z)$ by replacing any atomic formula $kx + t(z) \in U_{p,l}$ appearing in $\theta_p(x, z)$ with $kx + t^h(z) \in U_{p,l}$. Now for $p \leqslant B$, let l_p be the largest value of l occurring in an atomic formula in $\theta_p(x, z)$. Set

$$D = \prod_{p \leqslant B} p^{l_p}.$$

Obtain $a_p \in \mathbb{Z}$ such that $\theta_p(a_p, c)$ holds in $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. Equivalently, we have $\theta^h_p(ha_p, hc)$ holds in

$(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. By the Chinese remainder theorem, we get r in $\{0, \dots, D-1\}$ such that

$$r \equiv_{p^{l_p}} a_p \quad \text{for all } p \leqslant B.$$

We check that r is as desired. Suppose $a \in \mathbb{Z}$ is such that $a \equiv_D hr$. By construction, if $p \leqslant B$, $l \leqslant l_p$, and $kx + t(z) \in U_{p,l}$ is any atomic formula, then $ka + t^h(hc) \in U^{\mathbb{Z}}_{p,l}$ if and only if $k(ha_p) + t^h(hc) \in U^{\mathbb{Z}}_{p,l}$. It follows that $\theta^h_p(a, hc)$ is equivalent to $\theta^h_p(ha_p, hc)$ in $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. Thus $\theta^h_p(a, hc)$ holds for all $p \leqslant B$. ☐

Towards showing that the structures of interest are generic, the key number-theoretic ingredient we need is the following result:

Lemma 2.15. *Let $\psi(x, z, z')$ be a special formula and $\psi(x, c, c')$ a nontrivial $\mathbb{Z}$-system which is locally satisfiable in $\mathbb{Z}$. For $h > 0$, and $s, t \in \mathbb{Q}$ with $s < t$, set*

$$\Psi^h(hs, ht) = \{a \in \mathbb{Z} : \psi^h(a, hc, hc') \text{ holds and } hs < a < ht\}.$$

Then there exists $N \in \mathbb{N}^{>0}$, $\varepsilon \in (0, 1)$, and $C \in \mathbb{R}$ such that for all $h > 0$ with $\gcd(h, N!) = 1$ and $s, t \in \mathbb{Q}$ with $s < t$, we have that

$$|\Psi^h(hs, ht)| \geqslant \varepsilon h(t - s) - \left(\sum_{i=1}^{n} \sqrt{|hks + hc_i|} + \sqrt{|hkt + hc_i|} \right) + C.$$

Proof. Throughout this proof, let $\psi(x, z, z')$, $\psi(x, c, c')$, and $\Psi^h(hs, ht)$ be as stated. We first make a number of observations. Suppose $\psi(x, z, z')$ corresponds to the parameter choice (k, m, Θ) and has a boundary B, and $\psi_p(x, z, z')$ is the associated p-condition of $\psi(x, z, z')$. Then $\psi^h(x, z, z')$ corresponds to the parameter choice (k, hm, Θ^h), and B is also a boundary of $\psi^h(x, z, z')$ by Lemma 2.13. Moreover $\psi^h_p(x, z, z')$ is the associated p-condition of $\psi^h(x, z, z')$. Since $\psi(x, c, c')$ is locally satisfiable in $\mathbb{Z}$, we can use Lemma 2.14 to fix $D \in \mathbb{N}^{>0}$ and $r \in \{0, \dots, D-1\}$ such that for each $h > 0$ with $\gcd(h, B!) = 1$, we have

$$a \equiv_D hr \text{ implies } \bigwedge_{p \leqslant B} \psi^h_p(a, hc, hc') \quad \text{for all } a \in \mathbb{Z}.$$

We note that D here is independent of the choice of h for all h with $\gcd(h, B!) = 1$.

We introduce a variant of $\Psi^h(hs, ht)$ which is needed in our estimation of $|\Psi^h(hs, ht)|$. Until the end of the proof, set $l_p = 2 + v_p(m)$. Fix primes $p_1, \dots, p_{n'}$ such that $p_1 > c_i$ for all $i \in \{1, \dots, n\}$, $p_1 > c'_{i'}$ for all $i' \in \{1, \dots, n'\}$ and

$$B < p_1 < \dots < p_{n'}.$$

For $M > p_{n'}$, $h > 0$ with $\gcd(h, B!) = 1$, define $\Psi^h_M(hs, ht)$ to be the set of $a \in \mathbb{Z}$ such that $hs < a < ht$ and

$$(a \equiv_D hr) \wedge \bigwedge_{B < p \leqslant M} \Big(\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p + v_p(h)}} 0) \Big) \wedge \bigwedge_{i'=1}^{n'} (ka + hc'_{i'} \notin P^{\mathbb{Z}}_{hm}).$$

It is not hard to see that $\Psi^h(hs, ht) \cap \{a \in \mathbb{Z} : a \equiv_D hr\} \subseteq \Psi^h_M(hs, ht)$, and the latter is intended to be an upper approximation of the former. The desired lower bound for $|\Psi^h(hs, ht)|$ will be obtained via a lower bound for $|\Psi^h_M(hs, ht)|$ and an upper bound for $|\Psi^h_M(hs, ht) \setminus \Psi^h(hs, ht)|$.

Now we work towards establishing a lower bound on $|\Psi^h_M(hs, ht))|$ in the case where $M > p_{n'}$, $h > 0$, and

$\gcd(h, M!) = 1$. The latter assumption implies in particular that $p^{l_p+v_p(h)} = p^{l_p}$ for all $p \leqslant M$. For $p > B$, we have that $p > |k|$ and so k is invertible mod p^{l_p}. Set

$$\Delta = \{p : B < p \leqslant M\} \setminus \{p_{i'} : 1 \leqslant i' \leqslant n'\}.$$

For $p \in \Delta$, as k is invertible mod p^{l_p}, there are at least $p^{l_p} - n$ (note we have $p > B > n$) choices of r_p in $\{0, \ldots, p^{l_p} - 1\}$ such that if $a \equiv_{p^{l_p}} r_p$, then

$$\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p}} 0).$$

Suppose $p = p_{i'}$ for some $i' \in \{1, \ldots, n'\}$. By the assumption that $\psi(x, c, c')$ is nontrivial, c has no common components with c'. Since $\gcd(h, M!) = 1$, h and p are coprime, and so the components of hc and hc' are pairwise distinct mod p^{l_p}. As k is invertible mod p^{l_p}, there is exactly one r_p in $\{0, \ldots, p^{l_p} - 1\}$ such that if $a \equiv_{p^{l_p}} r_p$, then

$$\bigwedge_{i=1}^{n} (ka + hc_i \not\equiv_{p^{l_p}} 0) \wedge (ka + hc'_{i'} \equiv_{p^{l_p}} 0) \text{ and consequently } ka + hc'_{i'} \notin P^{\mathbb{Z}}_{hm}.$$

Now it follows by the Chinese remainder theorem that,

$$|\Psi^h_M(hs, ht)| \geqslant \left\lfloor \frac{ht - hs}{D \prod_{B<p\leqslant M} p^{l_p}} \right\rfloor \prod_{p \in \Delta} \left(p^{l_p} - n\right).$$

Then it follows that,

$$|\Psi^h_M(hs, ht)| \geqslant \frac{ht - hs}{D} \prod_{p \leqslant p_{n'}} \frac{1}{p^{l_p}} \prod_{p > p_{n'}}^{\leqslant M} \left(1 - \frac{n}{p^{l_p}}\right) - \prod_{p \leqslant M} p^{l_p}.$$

Set

$$\varepsilon = \frac{1}{2D} \prod_{p \leqslant p_{n'}} \frac{1}{p^{l_p}} \prod_{p > p_{n'}} \left(1 - \frac{n}{p^{l_p}}\right).$$

Now as $l_p \geqslant 2$, for $U \in \mathbb{N}^{>0}$ with $U > \max\{p'_n, n^2\}$ we have that

$$\prod_{p > U} \left(1 - \frac{n}{p^{l_p}}\right) > \prod_{p > U} \left(1 - \frac{1}{p^{\frac{3}{2}}}\right).$$

Hence, it follows from Euler's product formula that $\varepsilon > 0$. We now have

$$|\Psi^h_M(hs, ht)| \geqslant 2\varepsilon(ht - hs) - \prod_{p \leqslant M} p^{l_p}.$$

We note that ε is independent of the choice of M and h, and will serve as the promised ε in the statement of the lemma.

Next we obtain a upper bound on $|\Psi^h_M(s, t) \setminus \Psi^h(s, t)|$ for $M > p_{n'}$ $h > 0$ and $\gcd(h, M!) = 1$. We arrange that $k > 0$ by replacing c by $-c$ and c' by $-c'$ if necessary. Note that an element $a \in \Psi^h_M(s, t) \setminus \Psi^h(s, t)$ must be such that

$$hks + hc_i < ka + hc_i < hkt + hc_i \quad \text{for all } i \in \{1,\dots,n\}$$

and $ka + hc_i$ is a multiple of p^{l_p} for some $p > M$ and $i \in \{1,\dots,n\}$. For each p and $i \in \{1,\dots,n\}$, the number of non-zero multiples of p^{l_p} in $(hks + hc_i, hkt + hc_i)$ is

$$\lfloor hk(t-s)p^{-l_p} \rfloor - 2, \text{ or } \lfloor hk(t-s)p^{-l_p} \rfloor - 1, \text{ or } \lfloor hk(t-s)p^{-l_p} \rfloor, \text{ or } \lfloor hk(t-s)p^{-l_p} \rfloor + 1.$$

In the last case, as $l_p \geqslant 2$ we moreover have

$$p^2 \leqslant |hks + hc_i| \quad \text{or} \quad p^2 \leqslant |hkt + hc_i|,$$

and so

$$p \leqslant \sqrt{|hks + hc_i|} + \sqrt{|hkt + hc_i|}.$$

As $l_p \geqslant 2$, we have $\lfloor hk(t-s)p^{-l_p} \rfloor \leqslant hk(t-s)p^{-2}$. Therefore we have that

$$|\Psi^h_M(s,t) \setminus \Psi^h(s,t)| \leqslant h(t-s) \sum_{p>M} \frac{nk}{p^2} + \left(\sum_{i=1}^{n} \sqrt{|hks + hc_i|} + \sqrt{|hkt + hc_i|} \right) + 1.$$

We now obtain N and C as in the statement of the lemma. Note that

$$\sum_{p>T} p^{-2} \leqslant \sum_{n>T} n^{-2} = O(T^{-1}).$$

Using this, we obtain $N \in \mathbb{N}^{>0}$ such that $N > p_{n'}$ and $\sum_{p>N} knp^{-2} < \varepsilon$ where ε is from the preceding paragraph. Set $C = -\prod_{p \leqslant N} p^{l_p} - 1$. Combining the estimations from the preceding two paragraphs for $M = N$ it is easy to see that ε, N, C are as desired. □

Remark 2.16. *The above weak lower bound is all we need for our purpose. We expect that a stronger estimate can be obtained using modifications of available techniques in the literature; see for example [48].*

Corollary 2.17. *For all $c \in \mathbb{Z}$, there is $a \in \mathbb{Z}$ such that*

$$a + c \in \mathrm{SF}^{\mathbb{Z}} \quad \textit{and} \quad a + c + 1 \in \mathrm{SF}^{\mathbb{Z}}.$$

Proof. We have that for all $c \in \mathbb{Z}$, $\psi(x,c) = (x + c \in \mathrm{SF}^{\mathbb{Z}}) \wedge (x + c + 1 \in \mathrm{SF}^{\mathbb{Z}})$ is a locally satisfiable $\mathbb{Z}$-system. Applying Lemma 2.15 for $h = 1$, $s = 0$, and t sufficiently large we see there is a solution $a \in \mathbb{Z}$ for $\psi(x,c)$. □

We next prove the main theorem of the section:

Theorem 2.18. *The $\mathrm{Sf}^*_{\mathbb{Z}}$-model $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$, the $\mathrm{Sf}^*_{\mathbb{Q}}$-model $(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$, and the $\mathrm{OSf}^*_{\mathbb{Q}}$-model $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ are generic.*

Proof. We get the first part of the theorem by applying Lemma 2.15 for $h = 1$, $s = 0$, and t sufficiently large. As the second part of the theorem follows easily from the third part, it will be enough to show that the $\mathrm{OSf}^*_{\mathbb{Q}}$-model $(\mathbb{Q}; <, \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ is generic. Throughout this proof, suppose $\psi(x, z, z)$ is a special formula and $\psi(x, c, c')$ is a $\mathbb{Q}$-system which is nontrivial and locally satisfiable in $\mathbb{Q}$. Our job is to show that the $\mathbb{Q}$-system $\psi(x, c, c')$ has a solution in the $\mathbb{Q}$-interval $(b, b')^{\mathbb{Q}}$ for an arbitrary choice of $b, b' \in \mathbb{Q}$ such that $b < b'$.

We first reduce to the special case where $\psi(x, c, c')$ is also a $\mathbb{Z}$-system which is nontrivial and locally satisfiable in $\mathbb{Z}$. Let B be the boundary of $\psi(x, z, z')$ and for each p, let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. Using the assumption that $\psi(x, c, c')$ is locally satisfiable $\mathbb{Q}$-system, for each $p < B$ we obtain $a_p \in \mathbb{Q}$ such that $\psi_p(a_p, c, c')$ holds. Let $h > 0$ be such that

$$hc \in \mathbb{Z}^n, hc' \in \mathbb{Z}^{n'} \text{ and } ha_p \in \mathbb{Z} \text{ for all } p < B.$$

Then by the choice of h , Lemma 2.11, and Lemma 2.13, the h-conjugate $\psi^h(x, hc, hc')$ of $\psi(x, c, c')$ is a $\mathbb{Z}$-system which is nontrivial and locally satisfiable in $\mathbb{Z}$. On the other hand, $\psi(x, c, c')$ has a solution in a interval $(b, b')^{\mathbb{Q}}$ if and only if

$$\psi^h(x, hc, hc') \text{ has a solution in } (hb, hb')^{\mathbb{Q}}.$$

Hence, by replacing $\psi(x, z, z')$ with $\psi^h(x, z, z')$, $\psi(x, c, c')$ with $\psi^h(x, hc, hc')$, and $(b, b')^{\mathbb{Q}}$ with $(hb, hb')^{\mathbb{Q}}$ if necessary we get the desired reduction.

We show $\psi(x, c, c')$ has a solution in the $\mathbb{Q}$-interval $(b, b')^{\mathbb{Q}}$ for the special case in the preceding paragraph. By an argument similar to the preceding paragraph, it suffices to show that for some $h \neq 0$, $\psi^h(x, hc, hc')$ has a solution in $(hb, hb')^{\mathbb{Q}}$. Applying Lemma 2.15 for $s = b$, $t = b'$, and h sufficiently large satisfying the condition of the lemma, we get the desired conclusion. $\square$

CHAPTER 3

The model theoretic consequences

In Section 3.1, we establish the model completeness and decidability assertions of Theorems 2.1, 2.2 2.3, 2.4, and the parts of these results that concern placing the structures on the combinatorial tameness spectrum are proven in Section 3.2.

3.1 Logical tameness

We will next prove that $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ admit quantifier elimination. We first need a technical lemma saying that modulo $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$, an arbitrary quantifier free formula $\phi(x, y)$ is not much more complicated than a special formula; recall that x always denotes a single variable.

Lemma 3.1. *Suppose $\varphi(x, y)$ is a quantifier-free L^*_{u}-formula. Then $\varphi(x, y)$ is equivalent modulo $\mathrm{Sf}^*_{\mathbb{Z}}$ to a disjunction of quantifier-free formulas of the form*

$$\rho(y) \wedge \varepsilon(x, y) \wedge \psi(x, t(y), t'(y))$$

where

(i) *$t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with length n and n' respectively;*

(ii) *$\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x, y)$ an equational condition, $\psi(x, z, z')$ a special formula.*

The corresponding statement with $\mathrm{Sf}^*_{\mathbb{Z}}$ replaced by $\mathrm{Sf}^*_{\mathbb{Q}}$ also holds.

Proof. Let $\varphi(x, y)$ be a quantifier-free L^*_{u}-formula. We will use the following disjunction observation several times in our proof: If $\varphi(x, y)$ is a finite disjunction of quantifier-free L^*_{u}-formulas and we have proven the desired statement for each of those, then the desired statement for $\varphi(x, y)$ follows. In particular, it allows us to assume that $\varphi(x, y)$ is the conjunction

$$\rho(y) \wedge \varepsilon(x, y) \wedge \bigwedge_p \eta_p(x, y) \wedge \bigwedge_{i=1}^{n} (k_i x + t_i(y) \in P_{m_i}) \wedge \bigwedge_{i=1}^{n'} (k'_i x + t'_i(y) \notin P_{m'_i})$$

where $\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x, y)$ is an equational condition, $k_1, \dots, k_n$ and $k'_1, \dots, k'_{n'}$ are in $\mathbb{Z} \setminus \{0\}$, $m_1, \dots, m_n$ and $m'_1, \dots, m'_{n'}$ are in $\mathbb{N}^{\geqslant 1}$, $t_1(y), \dots, t_n(y)$ and $t'_1(y), \dots, t'_n(y)$ are L^*_{u}-terms with variables in y, $\eta_p(x, y)$ is a p-condition for each p, and $\eta_p(x, y)$ is trivial for all but finitely many p.

We make further reductions to the form of $\varphi(x,y)$. Set $t(y) = (t_1(y),\dots,t_n(y))$ and $t'(y) = (t'_1(y),\dots,t'_{n'}(y))$. Using the disjunction observation and the fact that

$$(x + y_j \in P_1) \vee (x + y_j \notin P_1)$$

is a tautology for every component y_j of y, we can assume that either $x + y_j \in P_1$ or $x + y_j \notin P_1$ are among the conjuncts of $\varphi(x,y)$, and so y_j is among the components of $t(y)$ or $t'(y)$. Then we obtain for each prime p a p-condition $\theta_p(x,z,z')$ such that $\theta_p(x,t(y),t'(y))$ is logically equivalent to $\eta_p(x,y)$. Let $\xi(x,z,z')$ be the formula

$$\bigwedge_p \theta_p(x,z,z') \wedge \bigwedge_{i=1}^{n}(k_i x + z_i \in P_{m_i}) \wedge \bigwedge_{i=1}^{n'}(k'_i x + z'_i \notin P_{m'_i}).$$

Clearly, $\varphi(x,y)$ is equivalent to the formula $\rho(y) \wedge \varepsilon(x,y) \wedge \xi(x,t(y),t'(y))$, so we can assume that $\varphi(x,y)$ is the latter.

We need a small observation. For a p-condition $\theta_p(z)$ and $h \neq 0$, we will show that there is another p-condition $\eta_p(z)$ such that modulo $\mathrm{Sf}^*_{\mathbb{Z}}$ and $\mathrm{Sf}^*_{\mathbb{Q}}$,

$$\eta_p(z_1,\dots,z_{i-1},hz_i,z_{i+1},\dots,z_n) \text{ is equivalent to } \theta_p(z).$$

For the special case where $\theta_p(z)$ is $t(z) \in U_{p,l}$, the conclusion follows from Lemma 2.5(iii), Lemma 2.9(iii) and the fact that there is an L^*_{u}-term $t'(z)$ such that $t'(z,\dots,z_{i-1},hz_i,z_{i+1},\dots,z_n) = ht(z)$. The statement of the paragraph follows easily from this special case.

With $\varphi(x,y)$ as in the end of the second paragraph, we further reduce the main statement to the special case where there is $k \neq 0$ such that $k_i = k'_{i'} = k$ for all $i \in \{1,\dots,n\}$ and $i' \in \{1,\dots,n'\}$. Choose $k \neq 0$ to be a common multiple of $k_1,\dots,k_n$ and $k'_1,\dots k'_{n'}$. Then by Lemma 2.5(vi) and Lemma 2.9(iii), we have for each $i \in \{1,\dots,n\}$ that

$$k_i x + z_i \in P_{m_i} \text{ is equivalent to } (kx + kk_i^{-1}z_i \in P_{kk_i^{-1}m_i}) \text{ modulo either } \mathrm{Sf}^*_{\mathbb{Z}} \text{ or } \mathrm{Sf}^*_{\mathbb{Q}}.$$

We have a similar observation for k and $k'_{i'}$ with $i' \in \{1,\dots,n'\}$. The desired reduction easily follows from these observations and the preceding paragraph.

Continuing with the reduction in the preceding paragraph, we next arrange that there is $m > 0$ such that $m_i = m'_{i'} = m$ for all $i \in \{1,\dots,n\}$ and $i' \in \{1,\dots,n'\}$. Let m be a common multiple of $m_1,\dots,m_n$ and $m'_1,\dots m'_{n'}$. By Lemma 2.5(v, vi) and Lemma 2.9(iii), we have for $i \in \{1,\dots,n\}$ that modulo either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$

$$kx + z_i \in P_{m_i} \text{ is equivalent to } kx + z_i \in P_m \wedge \bigwedge_{p \mid \frac{m}{m_i}} kx + z_i \notin U_{p,2+v_p(m_i)}$$

and for $i' \in \{1,\dots,n'\}$ that modulo either $\mathrm{Sf}^*_{\mathbb{Z}}$ or $\mathrm{Sf}^*_{\mathbb{Q}}$

$$kx + z'_{i'} \notin P_{m'_{i'}} \text{ is equivalent to } kx + z'_{i'} \notin P_m \vee \bigvee_{p \mid \frac{m}{m'_{i'}}} kx + z'_{i'} \in U_{p,2+v_p(m'_{i'})}.$$

It follows that $\varphi(x,y)$ is equivalent to a disjunction of formulas of the form we are aiming for. The desired conclusion of the lemma follows from the disjunction observation. $\square$

Corollary 3.2. *Suppose $\varphi(x,y)$ is a quantifier-free L^*_{ou} formula. Then $\varphi(x,y)$ is equivalent modulo $\mathrm{OSf}^*_{\mathbb{Q}}$ to* *a disjunction of quantifier-free formulas of the form*

$$\rho(y) \wedge \lambda(x, y) \wedge \psi(x, t(y), t'(y))$$

where

(i) $t(y)$ *and* $t'(y)$ *are tuples of* L^*_{ou}*-terms with length* n *and* n'*respectively;*

(ii) $\rho(y)$ *is a quantifier-free* L^*_{ou}*-formula,* $\lambda(x, y)$ *an order condition,* $\psi(x, z, z')$ *a special formula.*

In the next lemma, we show a "local quantifier elimination" result.

Lemma 3.3. *If* $\varphi(x, z)$ *is a* p*-condition, then modulo either* $\mathrm{Sf}^*_{\mathbb{Z}}$ *or* $\mathrm{Sf}^*_{\mathbb{Q}}$*, the formula* $\exists x \varphi(x, z)$ *is equivalent to a* p*-condition* $\psi(z)$*.*

Proof. If $\varphi(x, z)$ is a p-condition, then by Lemma 2.5 (i), modulo $\mathrm{Sf}^*_{\mathbb{Z}}$, it is a boolean combination of atomic formulas of the form $kx + t(z) \in U_{p,l}$ where $t(z)$ is an L^*_{u}-term, and $l > 0$. Let l_p be the largest value of l occurring in such atomic formulas, and set

$$S = \{(m_1, \ldots, m_n) : 0 \leqslant m_i < p^{l_p} \text{ for each } i, \text{ and } (\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}) \models \exists x \varphi(x, m_1, \ldots, m_n)\}.$$

Then by Lemma 2.5 (i), modulo $\mathrm{Sf}^*_{\mathbb{Z}}$, $\exists x \varphi(x, z)$ is equivalent to the p-condition $\bigvee_{(m_1,\ldots,m_n)\in S}(\bigwedge_{i=1}^{n}(z_i \equiv_{p^{l_p}} m_i))$.

Now, we proceed to prove the statement for models of $\mathrm{Sf}^*_{\mathbb{Q}}$. Throughout the rest of the proof, suppose $\varphi(x, z)$ is a p-condition, k, k', l, l' are in $\mathbb{Z}$, and $t(z), t'(z)$ are L^*_{u}-terms. First, we consider the case where $\varphi(x, z)$ is a p-condition of the form $kx + t(z) \in U_{p,l}$. The case $k = 0$ is trivial. If $k \neq 0$, then $\exists x(kx + t(z) \in U_{p,l})$ is tautological modulo $\mathrm{Sf}^*_{\mathbb{Q}}$ following from (Q1) in the definition of $\mathrm{Sf}^*_{\mathbb{Q}}$ and Lemma 2.9(i).

We next consider the case where $\varphi(x, z)$ is a finite conjunction of p-conditions in $L^*_{\text{u}}(x, z)$ such that one of the conjuncts is $kx + t(z) \in U_{p,l}$ with $k \neq 0$ and the other conjuncts are either of the form $k'x + t'(z) \in U_{p,l'}$ or of the form $k'x + t'(z) \notin U_{p,l'}$ where we do allow l' to vary. It follows from Lemma 2.9(i) that if $k = k'$, $l \geqslant l'$, then

$$k'x + t'(z) \in U_{p,l'} \text{ if and only if } t(z) - t'(z) \in U_{p,l'}.$$

So we have means to replace conjuncts of $\varphi(x, z)$ by terms independent of the variable x. However, the above will not work if $k \neq k'$ or $l < l'$. By Lemma 2.9(iii), across models of $\mathrm{Sf}^*_{\mathbb{Q}}$, we have that

$$kx + t(z) \in U_{p,l} \text{ if and only if } hkx + ht(z) \in U_{p,l+v_p(h)} \quad \text{for all } h \neq 0.$$

From this observation, it is easy to see that we can resolve the issue of having $k \neq k'$, and moreover arrange that $l \geqslant 0$ which will be used in the next observation. By Lemma 2.9(i,ii), across models of $\mathrm{Sf}^*_{\mathbb{Q}}$, we have that

$$kx + t(z) \in U_{p,l} \text{ if and only if } \bigvee_{i=1}^{p^m} kz + t(z) + ip^l \in U_{p,l+m} \text{ for all } l \geqslant 0 \text{ and all } m.$$

Using the preceding two observations we resolve the issue of having $l < l'$. The statement of the lemma for this case then follows from the second paragraph.

We now prove the full lemma. It suffices to consider the case where $\varphi(x, z)$ is a conjunction of atomic formulas. In view of the preceding paragraph, we reduce further to the case where $\varphi(x,z)$ is of the form

$$\bigwedge_{i=1}^{m} kx + t_i(z) \notin U_{p,l_i}$$

We now show that $\exists x \varphi(x,z)$ is a tautology over $\mathrm{Sf}^*_{\mathbb{Q}}$ and thus complete the proof. Suppose $(G; \mathscr{U}^G, \mathscr{P}^G) \models \mathrm{Sf}^*_{\mathbb{Q}}$ and $c \in G^n$. It suffices to find $a \in G$ such that the p-condition $ka + t_i(c) \notin U^G_{p,l_i}$ holds for all $i \in \{1,\dots,m\}$. Without loss of generality, we assume that $t_1(c),\dots,t_{m'}(c)$ are not in $U^G_{p,l}$ for all l and that $t_{m'+1}(c),\dots,t_m(c)$ are in U^G_{p,l_0} for some l_0 such that $l_0 < l_i$ for all $i \in \{1,\dots,m\}$. Using 2.9(ii), choose a such that $ka \in U^G_{p,l_0-1} \setminus U^G_{p,l_0}$. It follows from Lemma 2.9(i) that a is as desired. □

Theorem 3.4. *The theories* $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, *and* $\mathrm{OSF}^*_{\mathbb{Q}}$ *admit quantifier elimination.*

Proof. As the three situations are very similar, we will only present here the proof that $\mathrm{OSF}^*_{\mathbb{Q}}$ admits quantifier elimination. The proof for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ are simpler as there is no ordering involved. Along the way we point out the necessary modifications needed to get the proof for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$. Fix $\mathrm{OSF}^*_{\mathbb{Q}}$-models $(G; <, \mathscr{U}^G, \mathscr{P}^G)$ and $(H; <, \mathscr{U}^H, \mathscr{P}^H)$ such that the latter is $|G|^+$-saturated. Suppose

$$f \text{ is a partial } L^*_{\mathrm{ou}}\text{-embedding from } (G; <, \mathscr{U}^G, \mathscr{P}^G) \text{ to } (H; <, \mathscr{U}^H, \mathscr{P}^H),$$

in other words, f is an L^*_{ou}-embedding of an L^*_{ou}-substructure of $(G; <, \mathscr{U}^G, \mathscr{P}^G)$ into $(H; <, \mathscr{U}^H, \mathscr{P}^H)$. By a standard test, it suffices to show that if $\mathrm{Domain}(f) \neq G$, then there is a partial L^*_{ou}-embedding from $(G; <, \mathscr{U}^G, \mathscr{P}^G)$ to $(H; <, \mathscr{U}^H, \mathscr{P}^H)$ which properly extends f. For the corresponding statements with $\mathrm{SF}^*_{\mathbb{Z}}$ or $\mathrm{SF}^*_{\mathbb{Q}}$, we need to consider instead $(G; \mathscr{U}^G, \mathscr{P}^G)$ and $(H; \mathscr{U}^H, \mathscr{P}^H)$ depending on the situation.

We remind the reader that our choice of language includes a symbol for additive inverse, and so $\mathrm{Domain}(f)$ is automatically a subgroup of G. Suppose $\mathrm{Domain}(f)$ is not a pure subgroup of G, that is, there is an element $\mathrm{Domain}(f)$ which is n-divisible in G but not n-divisible in $\mathrm{Domain}(f)$ for some $n > 0$. Then there is prime p and a in $G \setminus \mathrm{Domain}(f)$ such that $pa \in \mathrm{Domain}(f)$. Using divisibility of H, we get $b \in H$ such that $pb = f(pa)$. Let g be the extension of f given by

$$ka + a' \mapsto kb + f(a') \quad \text{ for } k \in \{1,\dots,p-1\} \text{ and } a' \in \mathrm{Domain}(f).$$

It is routine to check that g is an ordered group isomorphism from $\langle \mathrm{Domain}(f), a\rangle$ to $\langle \mathrm{Image}(f), b\rangle$. It is also easy to check using Lemma 2.9(iii) that $ka + a' \in U^G_{p',l}$ if and only if $kb + f(a') \in U^G_{p',l}$ and $ka + a' \in P^G_m$ if and only if $kb + f(a') \in U^G_m$ for all k, l, m, primes p', and $a' \in \mathrm{Domain}(f)$. Hence,

$$g \text{ is a partial } L^*_{\mathrm{ou}}\text{-embedding from } (G; <, \mathscr{U}^G, \mathscr{P}^G) \text{ to } (H; <, \mathscr{U}^H, \mathscr{P}^H).$$

Clearly, g properly extends f, so the desired conclusion follows. The proof for $\mathrm{SF}^*_{\mathbb{Q}}$ is the same but without the verification that the ordering is preserved. The situation for $\mathrm{SF}^*_{\mathbb{Z}}$ is slightly different as H is not divisible. However, for all primes p', $p'a$ is in $p'G = U^G_{p',1}$, and so $f(p'a)$ is in $U^H_{p',1} = p'H$. The proof proceeds similarly using 2.5(4-6).

The remaining case is when $\mathrm{Domain}(f) \neq G$ is a pure subgroup of G. Let a be in $G \setminus \mathrm{Domain}(f)$. We need to find b in $H \setminus \mathrm{Image}(f)$ such that

$$\mathrm{qftp}_{L^*_{\mathrm{ou}}}(a/\mathrm{Domain}(f)) = \mathrm{qftp}_{L^*_{\mathrm{ou}}}(b/\mathrm{Image}(f)).$$

By the fact that Domain(f) is pure in G, and Corollary 3.2, $\mathrm{qftp}_{L^*_{\mathrm{ou}}}(a \mid \mathrm{Domain}(f))$ is isolated by formulas of the form

$$\rho(b) \wedge \lambda(x, b) \wedge \psi(x, t(b), t'(b))$$

where $\rho(y)$ is a quantifier-free L^*_{ou}-formula, $\lambda(x, y)$ is an order condition, $\psi(x, z, z')$ a special formula, $t(y)$ and $t'(y)$ are tuples of L^*_{ou}-terms of suitable length, b is a tuple of elements of Domain(f) of suitable length, and $\psi(x, t(b), t'(b))$ is a nontrivial Domain(f)-system. As Domain(f) is a pure subgroup of G, we can moreover arrange that $\lambda(x, b)$ is simply the formula $b_1 < x < b_2$. Since f is an L^*_{ou}-embedding, $\rho(f(b))$ holds, $f(b_1) < f(b_2)$, and $\psi\big(x, t(f(b)), t'(f(b))\big)$ is a nontrivial Image(f)-system. Using the fact that $(H; <, \mathscr{U}^H, \mathscr{P}^H)$ is $|G|^+$-saturated, the problem reduces to showing that

$$\psi\Big(x, f\big(t(b)\big), f\big(t'(b)\big)\Big) \text{ has a solution in the interval } (f(b_1), f(b_2))^H.$$

As $\psi(x, t(b), t'(b))$ is satisfiable in G, it is locally satisfiable in G by Lemma 2.10. For each p, let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. By Lemma 3.3, for all p, the formula $\exists x \psi_p(x, z, z')$ is equivalent modulo $\mathrm{Sf}^*_{\mathbb{Q}}$ to a quantifier free formula in $L^*_{\mathrm{u}}(z, z')$. Hence, $\exists x \psi_p\Big(x, f\big(t(b)\big), f\big(t'(b)\big)\Big)$ holds in $(H; <, \mathscr{U}^H, \mathscr{P}^H)$ for all p. Thus,

$$\text{the Image}(f)\text{-system } \psi\Big(x, f\big(t(b)\big), f\big(t'(b)\big)\Big) \text{ is locally satisfiable in } H.$$

The desired conclusion follows from the genericity of $(H; <, \mathscr{U}^H, \mathscr{P}^H)$. The proofs for $\mathrm{SF}^*_{\mathbb{Z}}$ and $\mathrm{SF}^*_{\mathbb{Q}}$ are similar. However, we have there the formula $\bigwedge_{i=1}^k x \neq b_i$ with $k \leqslant |b|$ instead of the formula $b_1 < x < b_2$, Lemma 3.1 instead of Corollary 3.2, and the corresponding notion of genericity instead of the current one. ☐

Corollary 3.5. *The theory $\mathrm{SF}^*_{\mathbb{Z}}$ is a recursive axiomatization of $\mathrm{Th}(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$, and is therefore decidable. Similar statements hold for $\mathrm{SF}^*_{\mathbb{Q}}$ in relation to $\mathrm{Th}(\mathbb{Q}; \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$ and $\mathrm{OSF}^*_{\mathbb{Q}}$ in relation to $\mathrm{Th}(\mathbb{Q}; < \mathscr{U}^{\mathbb{Q}}, \mathscr{P}^{\mathbb{Q}})$.*

Proof. By Lemma 2.5(ii), the subgroup generated by 1 in an arbitrary model $(G; \mathscr{U}^G, \mathscr{P}^G)$ of $\mathrm{SF}^*_{\mathbb{Z}}$ is an isomorphic copy of $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$. Hence by Theorem 3.4, $\mathrm{SF}^*_{\mathbb{Z}}$ is complete, and on the other hand $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}}) \models \mathrm{SF}^*_{\mathbb{Z}}$ by Theorem 2.18. The first statement of the corollary follows. The justification of the second statement is obtained in a similar fashion. ☐

Proof of Theorem 2.1, part 1. We show that the L_{u}-theory of $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ is model complete and decidable. For all p, $l \geqslant 0$, $m > 0$, and all $a \in \mathbb{Z}$, we have the following:

1. $a \in U^{\mathbb{Z}}_{p,l}$ if and only there is $b \in \mathbb{Z}$ such that $p^l b = a$;
2. $a \notin U^{\mathbb{Z}}_{p,l}$ if and only if for some $i \in \{1, \dots, p^l - 1\}$, there is $b \in \mathbb{Z}$ such that $p^l b = a + i$;
3. $a \in P^{\mathbb{Z}}_m$ if and only if for some $d \mid m$, there is $b \in \mathbb{Z}$ such that $a = bd$ and $b \in \mathrm{SF}^{\mathbb{Z}}$;
4. $a \notin P^{\mathbb{Z}}_m$ if and only if for all $d \mid m$, either for some $i \in \{1, \dots, d-1\}$, there is $b \in \mathbb{Z}$ such that $db = a + i$ or there is $b \in \mathbb{Z}$ such that $a = bd$ and $b \notin \mathrm{SF}^{\mathbb{Z}}$.

As $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}}) \models \mathrm{SF}^*_{\mathbb{Z}}$, it then follows from Theorem 3.4 and the above observation that every 0-definable set in $(\mathbb{Z}, \mathrm{SF}^{\mathbb{Z}})$ is existentially 0-definable. Hence, the theory of $(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ is model complete. The decidability of $\mathrm{Th}(\mathbb{Z}; \mathrm{SF}^{\mathbb{Z}})$ is immediate from the preceding corollary. ☐

Lemma 3.6. *Suppose $a \in \mathbb{Q}$ has $v_p(a) < 0$. Then there is $\varepsilon \in \mathbb{Q}$ such that $v_p(\varepsilon) \geqslant 0$ and $a + \varepsilon \in \mathrm{SF}^{\mathbb{Q}}$.*

Proof. Suppose a is as stated. If $a \in \mathrm{SF}^{\mathbb{Q}}$ we can choose $\varepsilon = 0$, so suppose a is in $\mathbb{Q} \setminus \mathrm{SF}^{\mathbb{Q}}$. We can also arrange that $a > 0$. Then there are $m, n, k \in \mathbb{N}^{\geqslant 1}$ such that

$$a = \frac{m}{np^k}, \ (m,n) = 1, \ (m,p) = 1, \ \text{and } (n,p) = 1.$$

It suffices to show there is $b \in \mathbb{Z}$ such that $m + p^k b$ is a square-free integer as then

$$a + \frac{b}{n} = \frac{m + p^k b}{np^k} \in \mathrm{SF}^{\mathbb{Q}}.$$

For all prime l, $p^k b_l + m \notin U^{\mathbb{Q}}_{l,2}$ for $b_l = 0$ or 1. The conclusion then follows from the genericity of $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ as established in Theorem 2.18. $\square$

Corollary 3.7. *For all p and l, $U^{\mathbb{Q}}_{p,l}$ is universally 0-definable in $(\mathbb{Q}, \mathrm{SF}^{\mathbb{Q}})$.*

Proof. We will instead show that $\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l} = \{a : v_p(a) < l\}$ is existentially 0-definable for all p and l. As $\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l+n} = p^n(\mathbb{Q} \setminus U^{\mathbb{Q}}_{p,l})$ for all p, l, and n, it suffices to show the statement for $l = 0$. Fix a prime p. By the preceding lemma we have that for all a, $v_p(a) < 0$ if and only if

$$\text{there is } \varepsilon \text{ such that } v_p(\varepsilon) \geqslant 0, a + \varepsilon \in \mathrm{SF}^{\mathbb{Q}} \text{ and } v_p(a + \varepsilon) < 0.$$

We recall that $\{\varepsilon : v_p(\varepsilon) \geqslant 0\}$ is existentially 0-definable by Lemma 2.7. Also, for all $a' \in \mathrm{SF}^{\mathbb{Q}}$, we have that $v_p(a') < 0$ is equivalent to $p^2 a' \in \mathrm{SF}^{\mathbb{Q}}$. The conclusion hence follows. $\square$

Proof of Theorem 1.3 and 1.4, part 1. We show that the L_{u}-theory of $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$ and the L_{ou}-theory of $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$ are model complete and decidable. The proof is almost exactly the same as that of part 1 of Theorem 1.1. It follows from Lemma 2.7 and Corollary 3.7 that for all p and l, the sets $U^{\mathbb{Q}}_{p,l}$ are existentially and universally 0-definable in $(\mathbb{Q}; \mathrm{SF}^{\mathbb{Q}})$. For all m, $P^{\mathbb{Q}}_m = m\mathrm{SF}^{\mathbb{Q}}$ and $\mathbb{Q} \setminus P^{\mathbb{Q}}_m = m(\mathbb{Q} \setminus \mathrm{SF}^{\mathbb{Q}})$ are clearly existentially 0-definable. The conclusion follows. $\square$

Next, we will show that the L_{ou}-theory of $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ is bi-interpretable with arithmetic. The proof follow closely the arguments from [7]. In fact, we can slightly modify Corollary 3.9 to use essentially the same proof at the cost of replacing n^2 with $n^2 + n$.

Lemma 3.8. *Let $c_1, \dots, c_n$ be an increasing sequence of natural numbers, assume that for all primes p, there is a solution to the system of congruence inequations*

$$x + c_i \notin U^{\mathbb{Z}}_{p,2} \text{ for all } i \in \{1, \dots, n\}.$$

Then there is $a \in \mathbb{N}$ such that $a + c_1, \dots, a + c_n$ are consecutive square-free integers.

Proof. Suppose $c_1, \dots, c_n$ are as given. Let $c'_1, \dots, c'_{n'}$ be the listing in increasing order of elements in the set of $c \in \mathbb{N}$ such that $c_1 \leqslant c \leqslant c_n$ and $c \neq c_i$ for $i \in \{1, \dots, n\}$. The conclusion that there are infinitely many a such that

$$\bigwedge_{i=1}^{n} (a + c_i \in \mathrm{SF}^{\mathbb{Z}}) \wedge \bigwedge_{i=1}^{n'} (a + c'_i \notin \mathrm{SF}^{\mathbb{Z}})$$

follows from the assumptions about $c_1, \dots, c_n$ and the genericity of $(\mathbb{Z}; \mathscr{U}^{\mathbb{Z}}, \mathscr{P}^{\mathbb{Z}})$ as established in Theorem 2.18. $\square$

Corollary 3.9. *For all $n \in \mathbb{N}^{>0}$, there is $a \in \mathbb{N}$ such that $a+1, a+4, \dots, a+n^2$ are consecutive square-free integers .*

Proof. For each p, we can obtain $a \in \{1, 2, \dots, p^2 - 1\}$ such that

$$a \not\equiv_{p^2} -m^2 \text{ for all } m.$$

Hence, for any given $n > 0$ and p, the p-condition $\bigwedge_{i=1}^{n}(x + i^2 \notin U^{\mathbb{Z}}_{p,2})$ has a solution. The result now follows immediately from the preceding lemma. $\square$

Proof of Theorem 2.2. It suffices to show that $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ interprets multiplication on $\mathbb{N}$. Let T be the set of $(a, b) \in \mathbb{N}^2$ such that for some $n \in \mathbb{N}^{\geqslant 1}$,

$$b = a + n^2 \text{ and } a+1, a+4, \dots, a+n^2 \text{ are consecutive square-free integers.}$$

The set T is definable in $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ as $(a, b) \in T$ and $b \neq a + 1$ if and only if $a + 4 \leqslant b$, $a + 1$ and $a + 4$ are consecutive square-free integers, b is square-free, and whenever c, d, and e are consecutive square-free integers with $a < c < d < e \leqslant b$, we have that

$$(e - d) - (d - c) = 2.$$

Let S be the set $\{n^2 : n \in \mathbb{N}\}$. If $c = 0$ or there are a, b such that $(a, b) \in T$ and $b - a = c$, then $c = n^2$ for some n. Conversely, if $c = n^2$, then either $c = 0$ or by Corollary 3.9,

$$\text{there is } (a, b) \in T \text{ with } b - a = c.$$

Therefore, S is definable in $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$. The map $n \mapsto n^2$ in $\mathbb{N}$ is definable in $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$ as $b = a^2$ if and only if $b \in S$ and whenever $c \in S$ is such that $c > b$ and b, c are consecutive in S, we have that $c - b = 2a + 1$. Finally, $c = ba$ if and only if $2c = (b + a)^2 - b^2 - a^2$. Thus, multiplication on $\mathbb{N}$ is definable in $(\mathbb{Z}; <, \mathrm{SF}^{\mathbb{Z}})$. $\square$

3.2 Combinatorial tameness

As the theories $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ are complete, it is convenient to work in the so-called monster models, that is, models which are very saturated and homogeneous. Until the end of the section, let $(\mathbb{G}; \mathscr{U}^{\mathbb{G}}, \mathscr{P}^{\mathbb{G}})$ be a monster model of either $\mathrm{SF}^*_{\mathbb{Z}}$ or $\mathrm{SF}^*_{\mathbb{Q}}$ depending on the situation. In the latter case, we suppose $(\mathbb{G}; <, \mathscr{U}^{\mathbb{G}}, \mathscr{P}^{\mathbb{G}})$ is a monster model of $\mathrm{OSF}^*_{\mathbb{Q}}$. We assume that κ, A and I have small cardinalities compared to $\mathbb{G}$.

Our general strategy to prove the tameness of $\mathrm{SF}^*_{\mathbb{Z}}$, $\mathrm{SF}^*_{\mathbb{Q}}$, and $\mathrm{OSF}^*_{\mathbb{Q}}$ is to link them to the corresponding "local" facts. The next lemma says that $\mathrm{SF}^*_{\mathbb{Z}}$ is "locally" supersimple of U-rank 1.

Lemma 3.10. *Suppose $(\mathbb{G}; \mathscr{U}^{\mathbb{G}}, \mathscr{P}^{\mathbb{G}}) \models \mathrm{SF}^*_{\mathbb{Z}}$, $\theta_p(x, y)$ is a consistent p-condition, and b is in $\mathbb{G}^{|y|}$. Then $\theta_p(x, b)$ does not divide over any base set $A \subseteq \mathbb{G}$.*

Proof. Recall that every every p-condition is equivalent modulo $\mathrm{SF}^*_{\mathbb{Z}}$ to a formula in the language L of groups, and the reduct of $\mathrm{SF}^*_{\mathbb{Z}}$ to L is simply $\mathrm{Th}(\mathbb{Z})$. Hence, the desired conclusion is an immediate consequence of the well-known fact that $\mathrm{Th}(\mathbb{Z})$ is superstable of U-rank 1 [8]; see for example . $\square$

Proof of Theorem 2.1, part 2. We first show that $\mathrm{Th}(\mathbb{Z};\mathrm{SF}^{\mathbb{Z}})$ is supersimple of U-rank 1; see [42, p. 36] for a definition of U-rank or SU-rank. By the fact that $(\mathbb{Z};\mathrm{SF}^{\mathbb{Z}})$ has the same definable sets as $(\mathbb{Z};\mathscr{U}^{\mathbb{Z}},\mathscr{P}^{\mathbb{Z}})$ and Corollary 3.5, we can replace $\mathrm{Th}(\mathbb{Z};\mathrm{SF}^{\mathbb{Z}})$ with $\mathrm{SF}^*_{\mathbb{Z}}$. Suppose $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})\models\mathrm{SF}^*_{\mathbb{Z}}$. Our job is to show that every $L^*_{\mathrm{u}}(\mathbb{G})$-formula $\varphi(x,b)$ which forks over a small subset A of $\mathbb{G}$ must define a finite set in $\mathbb{G}$. We can easily reduce to the case that $\varphi(x,b)$ divides over A. Moreover, we can assume that $\varphi(x,b)$ is quantifier free by Theorem 3.4 which states that $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$ admits quantifier elimination. Using Lemma 3.1, we can also arrange that $\varphi(x,b)$ has the form

$$\rho(b)\wedge\varepsilon(x,b)\wedge\psi(x,t(b),t'(b))$$

where $\rho(y)$ is a quantifier-free formula, $\varepsilon(x,y)$ is an equational condition, $t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with length n and n' respectively, and $\psi(x,z,z')$ is a special formula.

Suppose to the contrary that $\varphi(x,b)$ divides over A but $\varphi(x,b)$ defines an infinite set in $\mathbb{G}$. From the first assumption, we get an infinite ordering I and a family $(\sigma_i)_{i\in I}$ of L^*_{u}-automorphisms of $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$ such that $(\sigma_i(b))_{i\in I}$ is indiscernible over A and $\bigwedge_{i\in I}\varphi(x,\sigma_i(b))$ is inconsistent. As $\varphi(x,b)$ defines an infinite set in $\mathbb{G}$, we get from the second assumption that $\rho(b)$ holds in $\mathbb{G}$, $\varepsilon(x,b)$ defines a cofinite set in $\mathbb{G}$, and $\psi(x,t(b),t'(b))$ defines an infinite hence non-empty set in $\mathbb{G}$. As $(\sigma_i(b))_{i\in I}$ is indiscernible, we have that $\rho(\sigma_i(b))$ holds in $\mathbb{G}$ and $\varepsilon(x,\sigma_i(b))$ defines a cofinite set in $\mathbb{G}$ for all $i\in I$. Using the saturation of $\mathbb{G}$, we get a finite set $\Delta\subseteq I$ such that

$$\theta_\Delta(x):=\bigwedge_{i\in\Delta}\psi\big(x,t(\sigma_i(b)),t'(\sigma_i(b))\big)\text{ defines a finite set in }\mathbb{G}.$$

As $\theta_\Delta(x)$ is a conjunction of $\mathbb{G}$-systems given by the same special formula, it is easy to see that $\theta_\Delta(x)$ is also a $\mathbb{G}$-system.

We will show that $\theta_\Delta(x)$ defines an infinite set and thus obtain the desired contradiction. As $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$ is a model of $\mathrm{SF}^*_{\mathbb{Z}}$ and hence generic, it suffices to show that $\theta_\Delta(x)$ is non-trivial and locally satisfiable. As $\varphi(x,b)$ is consistent, $t(b)$ has no common components with $t'(b)$. The assumption that $(\sigma_i(b))_{i\in I}$ is indiscernible gives us that $t(\sigma_i(b))$ has no common components with $t'(\sigma_j(b))$ for all i and j in I. It follows that $\theta_\Delta(x)$ is non-trivial. For each p, let $\psi_p(x,z,z')$ be the associated p-condition of $\psi(x,z,z')$. For all p, we have that $\psi_p(x,t(b),t(b'))$ defines a nonempty set and consequently by Lemma 3.10,

$$\bigwedge_{i\in\Delta}\psi_p\big(x,t(\sigma_i(b)),t'(\sigma_i(b))\big)\text{ defines a nonempty set in }\mathbb{G}.$$

We easily check that the above means $\theta_\Delta(x)$ is p-satisfiable for all p. Thus $\theta_\Delta(x)$ is locally satisfiable which completes our proof that $\mathrm{Th}(\mathbb{Z},\mathrm{SF}^{\mathbb{Z}})$ has U-rank 1.

We will next prove that $\mathrm{Th}(\mathbb{Z},\mathrm{SF}^{\mathbb{Z}})$ is k-independent for all $k>0$; see [19] for a definition of k-independence. The proof is almost the exact replica of the proof in [41] except the necessary modifications taken in the current paragraph. Suppose $l>0$, S is an arbitrary subset of $\{0,\dots,l-1\}$. Our first step is to show that there are $a,d\in\mathbb{N}$ such that for $t\in\{0,\dots,l-1\}$,

$$a+td\text{ is square-free}\quad\text{if and only if}\quad t\text{ is in }S.$$

Let $n=|S|$ and $n'=l-n$, and let $c\in\mathbb{Z}^n$ be the increasing listing of elements in S and $c'\in\mathbb{Z}^{n'}$ the increasing listing of elements in $\{0,\dots,l-1\}\setminus S$. Choose $d=(l!)^2$. We need to find a such that

$$\bigwedge_{i=1}^{n}(a+c_id\in \mathrm{SF}^{\mathbb{Z}})\wedge\bigwedge_{i=1}^{n'}(a+c_i'd\notin \mathrm{SF}^{\mathbb{Z}}).$$

For $p\leqslant l$, if $a_p\notin p^2\mathbb{Z}=U^{\mathbb{Z}}_{p,2}$, then $a_p+c_id\notin p^2\mathbb{Z}$ for all $i\in\{1,\dots,n\}$. For $p>l$, it is easy to see that $0+c_id\notin p^2\mathbb{Z}$ for all $i\in\{1,\dots,n\}$. The desired conclusion follows from the genericity of $(\mathbb{Z};\mathscr{U}^{\mathbb{Z}},\mathscr{P}^{\mathbb{Z}})$.

Fix $k>0$. We construct an explicit L_{u}-formula which witnesses the k-independence of $\mathrm{Th}(\mathbb{Z},\mathrm{SF}^{\mathbb{Z}})$. Let $y=(y_0,\dots,y_{k-1})$ and let $\varphi(x,y)$ be a quantifier-free L^*_{u}-formula such that for all $a\in\mathbb{Z}$ and $b\in\mathbb{Z}^k$,

$$\varphi(a,b)\ \text{ if and only if }\ a+b_0+\dots+b_{k-1}\in\mathrm{SF}^{\mathbb{Z}}\quad\text{ where } b=(b_0,\dots,b_{k-1}).$$

We will show that for any given $n>0$, there are families $(a_\Delta)_{\Delta\subseteq\{0,\dots,n-1\}^k}$ and $(b_{ij})_{0\leqslant i<k,0\leqslant j<n}$ of integers such that

$$\varphi(a_\Delta,b_{0,j_0},\dots,b_{k-1,j_{k-1}})\ \text{ if and only if }\ (j_0,\dots,j_{k-1})\in\Delta.$$

Let $f:\mathscr{P}(\{0,\dots,n-1\}^k)\to\{0,\dots,2^{(n^k)}-1\}$ be an arbitrary bijection. Let g be the bijection from $\{0,\dots,n-1\}^k$ to $\{0,\dots,n^k-1\}$ such that if b and b' are in $\{0,\dots,n-1\}^k$ and $b<_{\mathrm{lex}}b'$, then $g(b)<g(b')$. More explicitly, we have

$$g(j_0,\dots,j_{k-1})=j_0n^{k-1}+j_1n^{k-2}+\dots+j_{k-1}\text{ for }(j_0,\dots,j_{k-1})\in\{0,\dots,n-1\}^k.$$

It follows from the preceding paragraph that we can find an arithmetic progression $(c_i)_{i\in\{0,\dots,n^k2^{(n^k)}-1\}}$ such that for all $\Delta\subseteq\{0,\dots,n-1\}^k$ and $(j_0,\dots,j_{k-1})$ in $\{0,\dots,n-1\}^k$, we have that

$$c_{f(\Delta)n^k+g(j_0,\dots,j_{k-1})}\in\mathrm{SF}^{\mathbb{Z}}\ \text{ if and only if }\ (j_0,\dots,j_{k-1})\in\Delta.$$

Suppose $d=c_1-c_0$. Set $b_{ij}=djn^{k-i-1}$ for $i\in\{0,\dots,k-1\}$ and $j\in\{0,\dots,n-1\}$, and set $a_\Delta=c_{f(\Delta)n^k}$ for $\Delta\subseteq\{0,\dots,n-1\}^k$. We have

$$c_{f(\Delta)n^k+g(j_0,\dots,j_{k-1})}=c_{f(\Delta)n^k}+dg(j_0,\dots,j_{k-1})=a_\Delta+b_{0,j_0}+\dots+b_{k-1,j_{k-1}}.$$

The conclusion thus follows. □

Lemma 3.11. *Every p-condition $\theta_p(x,y)$ is stable in $\mathrm{SF}^*_{\mathbb{Q}}$.*

Proof. Suppose $\theta_p(x,y)$ is as in the statement of the lemma. It is clear that if $\theta_p(x,y)$ does not contain the variable x, then it is stable. As stability is preserved under taking boolean combinations, we can reduce to the case where $\theta_p(x,y)$ is $kx+t(y)\in U_{p,l}$ with $k\neq0$. We note that for any b and b' in $\mathbb{G}^{|y|}$, the sets defined by $\theta_p(x,b)$ and $\theta_p(x,b')$ are either the same or disjoint. It follows easily that $\theta_p(x,y)$ does not have the order property; in other words, $\theta_p(x,y)$ is stable. Alternatively, the desired conclusion also follows from the fact that $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}})$ is an abelian structure and hence stable; see [62, p. 49] for the relevant definition and result. □

Proof of Theorem 2.3, part 2. We first show that $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ is simple. By the fact that $(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ has the same definable sets as $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$ and Corollary 3.5, we can replace $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ with $\mathrm{SF}^*_{\mathbb{Q}}$. Towards a contradiction, suppose that the latter is not simple. We obtain a formula $\varphi(x,y)$ witnessing the tree property of $\mathrm{SF}^*_{\mathbb{Q}}$; see [42, pp. 24-25] for the definition and proof that this is one of the equivalent characterizations of simplicity. We can arrange that $\varphi(x,y)$ is quantifier-free by Theorem 3.4. Recall that disjunction preserves simplicity of formulas; this can be shown directly as an exercise or can be seen immediately from the equivalence between (1) and (3) in [42, Lemma 2.4.1]. Hence using Lemma 3.1, we can arrange that $\varphi(x,y)$ is of the form

$$\rho(y) \wedge \varepsilon(x,y) \wedge \psi(x, t(y), t'(y))$$

where $\rho(y)$ is a quantifier-free L^*_{u}-formula, $\varepsilon(x,y)$ is an equational condition, $t(y)$ and $t'(y)$ are tuples of L^*_{u}-terms with lengths n and n' respectively, and $\psi(x,z,z')$ is a special formula. Let $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}}) \models \mathrm{SF}^*_{\mathbb{Q}}$. Then there is $b \in \mathbb{G}^k$ with $k = |y|$, an uncountable cardinal κ, and a tree $(\sigma_s)_{s\in\omega^{<\kappa}}$ of L^*_{u}-automorphisms of $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$ with the following properties:

1. for all $s \in \omega^{<\kappa}$, $\{\varphi(x, \sigma_{s\frown(i)}(b)) : i \in \omega\}$ is inconsistent;
2. for all $\hat{s} \in \omega^{\kappa}$, $\{\varphi(x, \sigma_{\hat{s}\restriction\alpha}(b)) : \alpha < \kappa\}$ is consistent;
3. for every $\alpha < \kappa$ and $s, s' \in \omega^{\alpha}$, $\mathrm{tp}\big((\sigma_{s\frown(i)}(b))_i\big) = \mathrm{tp}\big((\sigma_{s'\frown(i)}(b))_i\big)$.

More precisely, we can get b, κ, and $(\sigma_t)_{t\in\omega^{<\kappa}}$ satisfying (1) and (2) from the fact that $\varphi(x,y)$ witnesses the tree property of $\mathrm{SF}^*_{\mathbb{Q}}$, a standard Ramsey arguments, and the monstrosity of $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$. We can then arrange that (3) also holds using results in [43]; a direct argument is also straightforward.

We deduce the desired contradiction by showing that there is $s \in \omega^{<\kappa}$ such that $\{\varphi(x, \sigma_{s\frown(i)}(b)) : i \in \omega\}$ is consistent. From (1-3), we get for all $s \in \omega^{<\kappa}$ that $\rho(\sigma_s(b))$ holds and $\varepsilon(x, \sigma_s(b))$ defines a cofinite set. By montrosity of $\mathbb{G}$, it suffices to find $s \in \omega^{<\kappa}$ such that any finite conjunction of $\{\psi\big(x, t(\sigma_{s\frown(i)}(b)), t'(\sigma_{s\frown(i)}(b))\big) : i \in \omega\}$ defines an infinite set in $\mathbb{G}$. For $s \in \omega^{<\kappa}$ and a finite $\Delta \subseteq \omega$, set

$$\theta_{s,\Delta}(x) := \bigwedge_{i\in\Delta} \psi\big(x, t(\sigma_{s\frown(i)}(b)), t'(\sigma_{s\frown(i)}(b))\big).$$

As κ is uncountable, to ensure the desired $s \in \omega^{<\kappa}$ exists, it suffices to show for fixed Δ that for all but countably many $\alpha < \kappa$ and all $s \in \omega^{\alpha}$, the formula $\theta_{s,\Delta}(x)$ defines an infinite set in $\mathbb{G}$.

Note that $\theta_{s,\Delta}(x)$ is a conjunction of $\mathbb{G}$-systems given by the same special formula, so $\theta_{s,\Delta}(x)$ is also a $\mathbb{G}$-system. By the genericity of $\mathrm{SF}^*_{\mathbb{Q}}$ established in Theorem 2.18, we need to check that for all but countably many $\alpha < \kappa$ and all $s \in \omega^{\alpha}$, the $\mathbb{G}$-system $\theta_{s,\Delta}(x)$ is nontrivial and locally satisfiable. Indeed, this implies that By (2), $\varphi(x,b)$ is consistent, and so is $\psi(x, t(b), t'(b))$. This implies in particular that $t(b)$ and $t'(b)$ have no common components. It then follows from (3) that for $s \in \omega^{<\kappa}$ and $i, j \in \omega$,

$$t(\sigma_{s\frown(i)}(b)) \text{ and } t'(\sigma_{s\frown(j)}(b)) \text{ have no common elements .}$$

Hence, $\theta_{s,\Delta}(x)$ is nontrivial for all $s \in \omega^{<\kappa}$. Let $\psi_p(x,z,z')$ be the associated p-condition of $\psi(x,z,z')$. We then get from (2) that $\{\psi_p(x, t(\sigma_{\hat{s}\restriction\alpha}(b)), t'(\sigma_{\hat{s}\restriction\alpha}(b))) : \alpha < \kappa\}$ is consistent for all $\hat{s} \in \omega^{\kappa}$. By Lemma 3.11, the formula $\psi_p(x, t(y), t'(y))$ is stable and hence does not witness the tree property. It follows that for all but finitely many $\alpha < \kappa$ and all $s \in \omega^{\alpha}$, the set

$$\{\psi_p\big(x, t(\sigma_{s\frown(i)}(b)), t'(\sigma_{s\frown(i)}(b))\big) : i \in \omega\} \text{ is consistent.}$$

For such s, we have that $\theta_{s,\Delta}(x)$ is p-satisfiable. So for all but countably many $\alpha < \kappa$ and all $s \in \omega^{\alpha}$, $\theta_{s,\Delta}(x)$ is locally satisfiable which completes the proof that $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ is simple.

We next prove that $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ is not strong which implies that it is not supersimple; for the definition of strength and the relation to supersimplicity see [1]. Again, we can replace $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ by $\mathrm{SF}^*_{\mathbb{Q}}$ using Proposition 2.8 and Corollary 3.5. For each p, let $\varphi_p(x,y)$ with $|y|=1$ be the formula $x-y\in U_{p,0}$. For all p and i, set $b_{p,i}=p^{-i}$. We will show that $\big(\varphi_p(x,y),(b_{p,i})_{i\in\mathbb{N}}\big)$ forms an inp-pattern of infinite depth in $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$. For distinct i and j in $\mathbb{N}$, we have that $p^{-i}-p^{-j}\notin U^{\mathbb{Q}}_{p,0}$ which implies that $\varphi_p(x,b_{p,i})\wedge\varphi_p(x,b_{p,j})$ is inconsistent. On the other hand, if S is a finite set of primes, and $f:S\to\mathbb{N}$ is an arbitrary function, then for $a=\Sigma_{p\in S}b_{p,f(p)}$ we have that $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})\models\bigwedge_{p\in S}\varphi_p(a,b_{p,f(p)})$. The desired conclusion follows.

Finally, we note that $(\mathbb{Z};\mathscr{U}^{\mathbb{Z}},\mathscr{P}^{\mathbb{Z}})$ is a substructure of $(\mathbb{Q};\mathscr{U}^{\mathbb{Q}},\mathscr{P}^{\mathbb{Q}})$, the former theory admits quantifier elimination and has IP_k for all $k>0$. Therefore, the latter also has IP_k for all $k>0$. In fact, the construction in part 2 of the proof of Theorem 2.1 carries through. □

Lemma 3.12. *Any order-condition has* NIP *in* $\mathrm{OSF}^*_{\mathbb{Q}}$.

Proof. The statement immediately follows from the fact that every order condition is a formula in the language of ordered groups and the fact that the reduct of any model of $\mathrm{OSF}^*_{\mathbb{Q}}$ to this language is an ordered abelian group, which has NIP; see for example [40]. □

Proof of Theorem 2.4, part 2. In the proof of part 2 of Theorem 2.3, we have shown that $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ is not strong and is k-independent for all $k>0$, so the corresponding conclusions for $\mathrm{Th}(\mathbb{Q};<,\mathrm{SF}^{\mathbb{Q}})$ also follow. It remains to show that $\mathrm{Th}(\mathbb{Q};<,\mathrm{SF}^{\mathbb{Q}})$ has NTP_2. The proof is essentially the same as the proof that $\mathrm{Th}(\mathbb{Q};\mathrm{SF}^{\mathbb{Q}})$ is simple, but with extra complications coming from the ordering. By Proposition 2.8 and Corollary 3.5, we can replace $\mathrm{Th}(\mathbb{Q};<,\mathrm{SF}^{\mathbb{Q}})$ with $\mathrm{OSF}^*_{\mathbb{Q}}$. Towards a contradiction, assume that there is a formula $\varphi(x,y)$ witnessing TP_2 (see [18, pp. 700-701]). We can arrange that $\varphi(x,y)$ is quantifier-free by Theorem 3.4. Disjunctions of formulas with NTP_2 again have NTP_2[18, p. 701], so using Lemma 3.2 we can arrange that $\varphi(x,y)$ is of the form

$$\rho(y)\wedge\lambda(x,y)\wedge\psi(x,t(y),t'(y))$$

where $\rho(y)$ is a quantifier-free L^*_{ou}-formula, $\lambda(x,y)$ an order condition, $\psi(x,z,z')$ a special formula, and $t(y)$ and $t'(y)$ are tuples of L^*_{ou}-terms with length n and n' respectively. Then there is $b\in\mathbb{G}^k$ with $k=|y|$ and an array $(\sigma_{ij})_{i\in\omega,j\in\omega}$ of L^*_{ou}-automorphisms of $(\mathbb{G};<,\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$ with the following properties:

1. for all $i\in\omega$, $\{\varphi(x,\sigma_{ij}(b)):j\in\omega\}$ is inconsistent;
2. for all $f:\omega\to\omega$, $\{\varphi(x,\sigma_{if(i)}(b)):i\in\omega\}$ is consistent;
3. for all $i\in\omega$, $(\sigma_{ij}(b))_{j\in\omega}$ is indiscernible over $\{\sigma_{i'j}(b):i'\in\omega,i'\neq i,j\in\omega\}$;
4. the sequence of "rows" $((\sigma_{ij}(b))_{j\in\omega})_{i\in\omega}$ is indiscernible.

We could get b, ω, and $(\sigma_{ij})_{i\in\omega,j\in\omega}$ as above from the definition of NTP_2, Ramsey arguments, and the monstrosity of $(\mathbb{G};\mathscr{U}^{\mathbb{G}},\mathscr{P}^{\mathbb{G}})$; see also [18, p. 697] for the type of argument we need to get (3).

We deduce that the set $\{\varphi(x,\sigma_{ij}b):j\in\omega\}$ is consistent for all $i\in\omega$, which is the desired contradiction. We get from (2) that $\rho(\sigma_{ij}b)$ holds for all $i\in\omega$ and $j\in\omega$. Hence, it suffices to show for all $i\in\omega$ that

$$\{\lambda(x,\sigma_{ij}b)\wedge\psi(x,t(\sigma_{ij}b),t'(\sigma_{ij}b)):j\in\omega\}\text{ is consistent.}$$

The order condition $\lambda(x, y)$ has NIP by Lemma 3.12, and so it has NTP$_2$. Using conditions (2-4), we get that

$$\{\lambda(x, \sigma_{ij}(b)) : j \in \omega\} \text{ is consistent for all } i \in \omega.$$

Hence, any finite conjunction from $\{\lambda(x, \sigma_{ij}(b)) : j \in \omega\}$ contains an open interval for all $i \in \omega$. For $i \in \omega$ and a finite $\Delta \subseteq \omega$, set

$$\theta_{i,\Delta}(x) := \bigwedge_{j \in \Delta} \psi\big(x, t(\sigma_{ij}(b)), t'(\sigma_{ij}(b))\big).$$

It suffices to show that $\theta_{i,\Delta}(x)$ defines a non-empty set in every non-empty $\mathbb{G}$-interval.

We have that $\theta_{i,\Delta}(x)$ is a conjunction of $\mathbb{G}$-system given by the same special formula, and so is again a $\mathbb{G}$-system. By the genericity of $\mathrm{OSF}^*_{\mathbb{Q}}$, the problem reduces to showing $\theta_{i,\Delta}(x)$ is nontrivial and locally satisfiable. By (2), $\varphi(x, b)$ is consistent, and so is $\psi(x, t(b), t'(b))$. This implies in particular that $t(b)$ and $t'(b)$ have no common components. It then follows from (3) that for $i \in \omega$ and distinct $j, j' \in \omega$,

$$t(\sigma_{ij}(b)) \text{ and } t'(\sigma_{ij'}(b)) \text{ have no common elements.}$$

Hence, $\theta_{i,\Delta}(x)$ is nontrivial for all $i \in \omega$. Let $\psi_p(x, z, z')$ be the associated p-condition of $\psi(x, z, z')$. We then get from (2) that $\{\psi_p(x, \sigma_{if(i)}(b)) : i \in \omega\}$ is consistent for all $f : \omega \to \omega$. By Lemma 3.11, the formula $\psi_p(x, t(y), t'(y))$ is stable and hence has NTP$_2$. It follows that for all but finitely many $i \in \omega$ the set

$$\{\psi_p\big(x, t(\sigma_{ij}(b)), t'(\sigma_{ij}(b))\big) : j \in \omega\} \text{ is consistent.}$$

Combining with (4), we get that $\theta_{i,\Delta}(x)$ is p-satisfiable for all p which completes the proof. □

Corollary 3.13. *The set $\mathbb{Z}$ is not definable in $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$.*

Proof. Towards a contradiction, suppose $\mathbb{Z}$ is definable in $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$. Then by Theorem 2.2, $(\mathbb{N}; +, \times, <, 0, 1)$ is interpretable in $(\mathbb{Q}; <, \mathrm{SF}^{\mathbb{Q}})$. It then follows from Theorem 2.4 that $(\mathbb{N}; +, \times, <, 0, 1)$ has NTP$_2$, but this is well-known to be false. □

Acknowledgements

We would like to thank Utkarsh Agrawal, William Balderrama, Alexander Dunn, Lou van den Dries, Allen Gehret, Itay Kaplan, and Erik Walsberg for dicussions and comments at various stages of the project.

Part II

On the Pila-Wilkie Theorem

CHAPTER 4

Background

This part of the thesis we provide a full exposition of the Pila-Wilkie Counting theorem following the original paper [53], but exploit cell decomposition more thoroughly to simplify the deduction from its main ingredients.

Chapter 4 consists of all the model theory required in the later chapters. In Chapter 5, we state the Counting theorem, and include our simplified proof modulo the two main ingredients.

Chapter 6 covers the first of these intermediate results, a Pila and Bombieri type interpolation, and has as such nothing to do with the theory of o-minimal structures. The other ingredient is an o-minimal analogue of the Yomdin-Gromov theorem, and is the technically most demanding. Taking ideas from Binyamini and Novikov [15] we do it more directly in Chapter 7 than in the original paper.

In Chapter 8, we obtain two generalizations of the Counting theorem due to Pila [51], one where instead of rational points we count points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ with a finite bound on its dimension, and one where instead we count points with coordinates that are algebraic of at most a given degree over $\mathbb{Q}$. The general approach is as in [51], but the technical details seem to us a bit simpler.

4.1 Notations and Conventions

Throughout this part of the thesis, $d, e, k, l, m, n \in \mathbb{N} = \{0, 1, 2, \dots\}$, and $\varepsilon, c, K \in \mathbb{R}^{>} := \{t \in \mathbb{R} : \ t > 0\}$. For $\alpha = (\alpha_1, \ \dots, \ \alpha_m) \in \mathbb{N}^m$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$, and given a field $\boldsymbol{k}$ (often $\boldsymbol{k} = \mathbb{R}$) we set $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ for the usual coordinate functions $x_1, \dots, x_m$ on $\boldsymbol{k}^m$, and likewise $a^\alpha := a_1^{\alpha_1} \cdots a_m^{\alpha_m}$ for any point $a = (a_1, \dots, a_m) \in \boldsymbol{k}^m$. Let $U \subseteq \mathbb{R}^m$ be open. For a function $f : U \to \mathbb{R}$ of class C^k and $\alpha \in \mathbb{N}^m$, $|\alpha| \leqslant k$,

$$f^{(\alpha)} \ := \ \frac{\partial^{|\alpha|}}{\partial x^\alpha} f$$

denotes the corresponding partial derivative of order α. We extend this to C^k-maps $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$,

$$f^{(\alpha)} \ := \ (f_1^{(\alpha)}, \dots, f_n^{(\alpha)}) : U \to \mathbb{R}^n$$

for α as before. This includes the case $m = 0$, where $\mathbb{R}^0$ has just one point and any map $f : U \to \mathbb{R}^n$ is of class C^k for all k, with $f^{(\alpha)} = f$ for the unique $\alpha \in \mathbb{N}^0$. For $a_1, \dots, a_n \in \mathbb{R}^{\geqslant} := \{t \in \mathbb{R} : \ t \geqslant 0\}$ the number $\max\{a_1, \dots, a_n\} \in \mathbb{R}^{\geqslant}$ equals 0 by convention if $n = 0$. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we set $|a| := \max\{|a_1|, \dots, |a_n|\}$ in $\mathbb{R}^{\geqslant}$; this conflicts with our notation $|\alpha|$ for $\alpha \in \mathbb{N}^n$, but in practice no confusion will arise. We also use these notational conventions when instead of $\mathbb{R}$ we have any o-minimal field with U and f definable in it. The rest of this chapter is devoted to basic facts concerning o-minimality, definability, and basic model theory.

4.2 O-minimal fields

O-minimality as a subject started in [25] and [55]. We give the key definitions in full detail, with examples, but state most results without proof. These proofs are in [28] as to general facts about o-minimal fields and the semialgebraic case, and in [11, 26, 32, 31, 58, 63] as to specific examples beyond the semialgebraic case.

This section covers all the model theoretic background required for this part of the thesis; with the exception of Section 7.4, which requires some model-theoretic compactness, *alias* saturation, which is fully exposed in Section 4.3.

Structures. Let M be a nonempty set. We consider the finite cartesian powers

$$M^n \ := \ \{a = (a_1, \dots, a_n) : \ a_1, \dots, a_n \in M\},$$

identifying in the usual way M^1 with M and M^{m+n} with $M^m \times M^n$. A *structure on* M is a sequence $\mathcal{S}$=$(\mathcal{S}_n)$ such that for all n,

1. $\mathcal{S}_n$ is a boolean algebra of subsets of M^n, that is, all $X \in \mathcal{S}_n$ are subsets of M^n, $M^n \in \mathcal{S}_n$, and for all $X, Y \in \mathcal{S}_n$ also $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{S}_n$.
2. For $n \geqslant 2$ and $1 \leqslant i < j \leqslant n$ the diagonal $\{a \in M^n : a_i = a_j\} \in \mathcal{S}_n$.
3. If $X \in \mathcal{S}_n$, then $M \times X \in \mathcal{S}_{n+1}$ and $X \times M \in \mathcal{S}_{n+1}$.
4. If $X \in \mathcal{S}_{n+1}$, then $\pi(X) \in \mathcal{S}_n$, where $\pi : M^{n+1} \to M^n$ is the projection map given by $\pi(a_1, \dots, a_n, a_{n+1}) = (a_1, \dots, a_n)$.

Let $\mathcal{S}$ be a structure on M. The definition of "structure" lacks symmetry, but in fact, if $X \in \mathcal{S}_n$ and σ is a permutation of $\{1, \dots, n\}$, then

$$\{\big(a_{\sigma(1)}, \dots, a_{\sigma(n)}\big) : \ a = (a_1, \dots, a_n) \in X\} \in \mathcal{S}_n.$$

Given a map $f : X \to M^n$ with $X \subseteq M^m$, we say that f *belongs to* $\mathcal{S}$ (or $\mathcal{S}$ *contains* f) if its graph, as a subset of M^{m+n}, belongs to $\mathcal{S}_{m+n}$; in that case $X \in \mathcal{S}_m$, $f(X) \in \mathcal{S}_n$, $f^{-1}(Y) \in \mathcal{S}_m$ for every $Y \in \mathcal{S}_n$, and the restriction $f|_{X_0} : X_0 \to R^n$ belongs to $\mathcal{S}$ for every $X_0 \subseteq X$ in $\mathcal{S}_m$. If $f : X \to M^n$ and $g : Y \to M^l$ belong to $\mathcal{S}$, where $X \subseteq M^m$ and $Y \subseteq M^n$, then the composition $g \circ f : X \cap f^{-1}(Y) \to M^l$ belongs to $\mathcal{S}$. The class of all structures on M is partially ordered by $\subseteq$:

$$\mathcal{S} \subseteq \mathcal{S}' \ :\Longleftrightarrow \ \mathcal{S}_n \subseteq \mathcal{S}'_n \text{ for all } n.$$

Any collection $\mathcal{C}$ of sets $X \subseteq M^n$ for various n gives rise to the *least* structure $\mathcal{S}$ on M that contains every $X \in \mathcal{C}$, where "least" is with respect to $\subseteq$.

Ordered fields. Let R be an ordered field: a field with a (strict) total order $<$ on its underlying set such that for all $a, b, c \in R$ we have

$$a < b \Rightarrow a + c < b + c, \qquad a < b, \ 0 < c \Rightarrow ac < bc.$$

The case to keep in mind is the field $\mathbb{R}$ of real numbers with its usual ordering, but in Chapter 7 we work in bigger ambient ordered fields, since results in that setting have consequences for $\mathbb{R}$ that are less easy to obtain otherwise. (This is where model theory comes into play.) The ordered field $\mathbb{Q}$ of rational numbers embeds uniquely into R as an ordered field. We use also the signs $\leqslant, >, \geqslant$ with the usual meaning derived from $<$, and set $R^{>} := \{a \in R :\ a > 0\}$, $R^{\geqslant} := \{a \in R :\ a \geqslant 0\}$. For $a \in R$ we set $|a| := a$ if $a \geqslant 0$ and $|a| := -a$ if $a \leqslant 0$. For $a = (a_1, \dots, a_n) \in R^n$ we set $|a| := \max(|a_1|, \dots, |a_n|) \in R^{\geqslant}$, which by convention equals 0 if $n = 0$.

An *interval* in R is a set $(a, b) := \{x \in R :\ a < x < b\}$, where $a, b \in R_\infty := R \cup \{-\infty, \infty\}$, $a < b$, where we extend $<$ to a total ordering on R_∞ by $-\infty < x < \infty$ for all $x \in R$. For $a \leqslant b$ in R_∞ we also set $[a, b] := \{x \in R_\infty :\ a \leqslant x \leqslant b\}$, but we do not call this an interval. We endow R with the order topology on its underlying set: it has the collection of intervals as a basis, and is a hausdorff topology. We also equip R^n with the corresponding product topology.

We call R *real closed* if $R^{>} = \{b^2 :\ 0 \neq b \in R\}$ and every polynomial $p(x) \in R[x]$ of odd degree has a zero in R. (This is equivalent to the field $R[i]$ with $i^2 = -1$ being algebraically closed.) In particular, the ordered field $\mathbb{R}$ of real numbers is real closed, and in some precise sense, all real closed fields have the same elementary properties as $\mathbb{R}$ (Tarski); we do not explicitly use that fact. Here and below $\mathbb{R}$ denotes the *ordered field* of real numbers, not just the set of real numbers.

O-Minimal Structures. Let R again be an ordered field. A *structure on* R is a structure $\mathcal{S}$ on its underlying set such that

(5) $\{(a, b) \in R^2 : a < b\} \in \mathcal{S}_2$ and the graphs of $+, \cdot : R^2 \to R$ lie in $\mathcal{S}_3$.

Let $\mathcal{S}$ be a structure on R with $\{a\} \in \mathcal{S}_1$ for all $a \in R$. Then every interval is in $\mathcal{S}_1$, for every polynomial $p \in R[x_1, \dots, x_n]$ the corresponding function $a \mapsto p(a) : R^n \to R$ belongs to $\mathcal{S}$, and so $\mathcal{S}_n$ contains the sets

$$\{a \in R^n :\ p(a) = 0\} \text{ and } \{a \in R^n :\ p(a) > 0\}.$$

For real closed R, a *semialgebraic subset of* R^n is a finite union of sets

$$\{a \in R^n : p(a) = 0,\ q_1(a) > 0, \dots, q_m(a) > 0\}, \quad (p, q_1, \dots, q_m \in R[x_1, \dots, x_n]),$$

and setting $\mathcal{S}_n := \{\text{semialgebraic subsets of } R^n\}$ gives by the Tarski-Seidenberg theorem a structure $\mathcal{S} = (\mathcal{S}_n)$ on the ordered field R. This is the least structure on R containing $\{a\}$ for all $a \in R$. In this case $\mathcal{S}_1$ contains exactly the finite unions of the sets $\{a\}$ with $a \in R$ and intervals. This fact about $\mathcal{S}_1$ is a surprisingly strong minimality property of $\mathcal{S}$ which we now axiomatize:

An *o-minimal structure on* R is a structure on the ordered field R such that

(6) $\{a\} \in \mathcal{S}_1$ for every $a \in R$ and every element of $\mathcal{S}_1$ is a finite union of one-element subsets of R and intervals.

One can show that R must be real closed if there is an o-minimal structure on it. The theory of o-minimal structures is a wide ranging generalization of the older subject of semialgebraic sets, and much of the tame properties of semialgebraic sets go through for the sets belonging to an o-minimal structure, as we shall see. The significance of o-minimality for applications is largely due to the fact that there are interesting o-minimal structures on $\mathbb{R}$ beyond its structure of semialgebraic sets. The important examples below are by way of illustration; the general facts about o-minimal structures that we focus on in this section do not depend on the nontrivial theorems that establish the o-minimality of these examples.

Terminology: an *o-minimal field* is an ordered field equipped with an o-minimal structure on it (and this ordered field is then real closed). We let $\mathcal{S}_{\text{alg}}$ be the o-minimal structure on $\mathbb{R}$ consisting of the semialgebraic subsets of $\mathbb{R}^n$, for all n. The first examples of o-minimal fields beyond the semialgebraic case are:

(i) $\mathbb{R}_{\text{an}}$: this is $\mathbb{R}$ equipped with the smallest structure $\mathcal{S}_{\text{an}}$ on it that contains every $f : [-1,1]^n \to \mathbb{R}$ that extends to a real analytic function $U \to \mathbb{R}$ on some open neighborhood $U \subseteq \mathbb{R}^n$ of $[-1,1]^n$, for $n = 0,1,2,\dots$. A set $X \subseteq \mathbb{R}^n$ belongs to $\mathcal{S}_{\text{an}}$ iff X is subanalytic in the larger (compact) real analytic manifold $\mathbb{P}(\mathbb{R})^n$, where $\mathbb{P}(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is the real projective line. The study of $\mathbb{R}_{\text{an}}$ is essentially the theory of subanalytic sets due to Hironaka and Gabrielov: see [11, 26].

(ii) $\mathbb{R}_{\text{exp}}$: this is $\mathbb{R}$ with the smallest structure $\mathcal{S}_{\text{exp}}$ on it containing $\{r\}$ for all $r \in \mathbb{R}$, and the function $\exp : \mathbb{R} \to \mathbb{R}$, $\exp(r) := \mathrm{e}^r$. A set $X \subseteq \mathbb{R}^m$ belongs to $\mathcal{S}_{\text{exp}}$ iff $X = \pi\big(\{a \in \mathbb{R}^n : \ P(a, \mathrm{e}^a) = 0\}\big)$ for some $n \geqslant m$ and some polynomial $P \in \mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n]$, where $\mathrm{e}^a := (\mathrm{e}^{a_1},\dots,\mathrm{e}^{a_n})$ and $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is given by $\pi(a) = (a_1,\dots,a_m)$ for $a = (a_1,\dots,a_n) \in \mathbb{R}^n$. This characterization of $\mathcal{S}_{\text{exp}}$ is part of Wilkie's theorem in [63].

(iii) For applications in arithmetic algebraic geometry it is important that we can amalgamate (i) and (ii) into an o-minimal field $\mathbb{R}_{\text{an,exp}}$: this is $\mathbb{R}$ with the smallest structure $\mathcal{S}_{\text{an,exp}}$ on it such that $\mathcal{S}_{\text{an,exp}} \supseteq \mathcal{S}_{\text{an}}, \mathcal{S}_{\text{exp}}$. A characterization of $\mathcal{S}_{\text{an,exp}}$ in the style of (ii) is in [32], and a sharper one in [31] where also the description of $\mathcal{S}_{\text{an}}$ in (i) is improved.

In general, amalgamation as in Example (iii) does not preserve o-minimality: [58] describes two o-minimal structures $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathbb{R}$ for which the smallest structure $\mathcal{S}$ on $\mathbb{R}$ with $\mathcal{S} \supseteq \mathcal{S}_1, \mathcal{S}_2$ is not o-minimal.

As to the appearance of exponentiation in the examples above, [47] proves a striking dichotomy: for any o-minimal structure $\mathcal{S}$ on $\mathbb{R}$, either exp belongs to $\mathcal{S}$, or every function $\mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{S}$ is polynomially bounded, as $t \to \infty$. It is not known if there exists an o-minimal structures $\mathcal{S}$ on $\mathbb{R}$ with a function $\mathbb{R} \to \mathbb{R}$ belonging to it that grows faster, as $t \to \infty$, than any finite iterate of the exponential function.

One way that o-minimality can fail (very badly) for a structure $\mathcal{S}$ on $\mathbb{R}$ with $\{a\} \in \mathcal{S}_1$ for all $a \in \mathbb{R}$ is that $\mathbb{Z} \in \mathcal{S}_1$: one can show that then all closed subsets of all $\mathbb{R}^n$ belong to $\mathcal{S}$, and even the Lebesgue-measurability of certain sets in $\mathcal{S}$ cannot be settled without unorthodox set-theoretic axioms. In particular, the sine function on $\mathbb{R}$ cannot belong to any o-minimal structure on $\mathbb{R}$, although its restriction to any bounded interval belongs to the o-minimal structure $\mathcal{S}_{\text{an}}$ on $\mathbb{R}$.

Definable Sets. In the rest of this section we fix an o-minimal field R. Its underlying real closed ordered field is also denoted by R. For a set $X \subseteq R^m$ we call X *definable* if X belongs to the given o-minimal structure of R, and likewise for maps $X \to R^n$. (This use of the term "definable" has its origin in logic, for which see Section 4.3.) In case the given o-minimal structure on R consists just of the semialgebraic sets (in the sense of the real closed field R), we write *semialgebraic* in place of *definable*.

Topological notions like openness and continuity are with respect to the order topology on R and the corresponding product topology on each R^n. If $X \subseteq R^n$ is definable, then so are its closure $\mathrm{cl}(X)$ and its interior $\mathrm{int}(X)$ in $\mathbb{R}^n$. The definable homeomorphism $t \mapsto \frac{t}{1+|t|} \ : \ R \to (-1,1)$ extends to an order preserving bijection $R_\infty \to [-1,1]$ sending $-\infty$ to -1 and ∞ to 1, and we equip R_∞ with the (hausdorff) topology on it making this bijection into a homeomorphism.

Till further notice the results below are from [28, Chapter 3], where the o-minimal structures considered are more general, with just an underlying nonempty totally ordered set without least or greatest element and such that for any two distinct elements $a < b$ there is an x with $a < x < b$, no field operations being included.

Here is the key fact about univariate definable functions:

Theorem 4.1 (Monotonicity Theorem)**.** *Let $I = (a, b)$ be an interval and let $f : (a, b) \to R$ be definable. Then f has the following properties:*

(i) *there are points $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ such that on each subinterval (a_j, a_{j+1}) with $0 \leqslant j \leqslant n$ the function f is continuous, and either strictly decreasing, or constant, or strictly increasing.*

(ii) *if f is continuous and $f(p) < c < f(q)$ with $p < q$ in I, then $c = f(x)$ for some $x \in (p, q)$.* (*Intermediate Value Property.*)

(iii) $\lim_{t\downarrow a} f(t)$ *and* $\lim_{t\uparrow b} f(t)$ *exists in* R_∞.

Of course, the intermediate value property (ii) is automatic when the underlying ordered field is $\mathbb{R}$ and then requires no definability assumption. In the o-minimal setting, and certainly outside the familiar real environment, we confine attention to definable objects. For example, the correct analogue of "connected" is as follows: a definable set $X \subseteq R^m$ is said to be *definably connected* if there are no disjoint nonempty definable open subsets X_0, X_1 of X with $X = X_0 \cup X_1$. For such X and any definable continuous map $f : X \to R^n$, the image $f(X) \subseteq R^n$ is also definably connected. Intervals are definably connected.

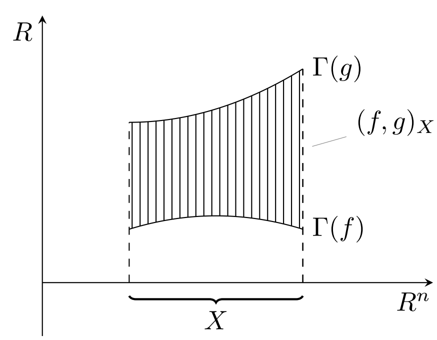
Cells. Towards partitioning an arbitrary definable set $X \subseteq R^n$ into finitely many nice pieces we introduce *cells*. These are definably connected sets of a form that makes them suited to proofs by induction (on n, for cells in R^n). First some notation. Let $X \subseteq R^n$ be definable. Set

$$\begin{aligned} C(X) &:= \{f : X \to R : \ f \text{ is definable and continuous}\}, \\ C_\infty(X) &:= C(X) \cup \{-\infty, \infty\}, \end{aligned}$$

where $-\infty$ and ∞ are viewed as constant functions on X. Let $f, g \in C_\infty(X)$, and suppose $f < g$, that is, $f(x) < g(x)$ for all $x \in X$. Then we set

$$(f, g) \ = \ (f, g)_X \ := \ \{(x, r) \in X \times R : \ f(x) < r < g(x)\},$$

so $(f, g) \subseteq R^{n+1}$ is definable; see next picture.



Let $n \geqslant 1$ and $(i_1, \ldots, i_n)$ a sequence of zeros and ones. An $(i_1, \ldots, i_n)$-cell is a definable subset of R^n obtained via the following recursion:

(i) Case $n = 1$: a (0)-cell is a one-element subset of R, a (1)-cell is an interval;

(ii) An $(i_1, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$ on an $(i_1, \ldots, i_n)$-cell X; an $(i_1, \ldots, i_n, 1)$-cell is a set $(f, g)_X$ with $f, g \in C_\infty(X)$, $f < g$, and X an $(i_1, \ldots, i_n)$-cell.

A *cell in* R^n is an $\boldsymbol{i}$-cell, for some (necessarily unique) $\boldsymbol{i} = (i_1, \ldots, i_n) \in \{0,1\}^n$. A cell in R^n is definably connected, and locally closed (open in its closure in R^n). A cell in R^n is open in R^n (and called an *open cell*) iff it is a $(1, \ldots, 1)$-cell.

Important in Section 5.2 and Chapter 8 is that every cell is homeomorphic under a coordinate projection to an open cell. In detail, let $C \subseteq R^n$ be an $\boldsymbol{i}$-cell, $\boldsymbol{i} = (i_1, \ldots, i_n)$. Let $\lambda(1) < \cdots < \lambda(k)$ be the indices $\lambda \in \{1, \ldots, n\}$ with $i_\lambda = 1$, and consider the (definable) coordinate projection $p_{\boldsymbol{i}} : R^n \to R^k$ given by

$$p_{\boldsymbol{i}}(x_1, \ldots, x_n) = (x_{\lambda(1)}, \ldots, x_{\lambda(k)}).$$

Then $p_{\boldsymbol{i}}$ maps C homeomorphically onto an open cell in R^k. We denote this open cell $p_{\boldsymbol{i}}(C)$ also by $p(C)$ and the homeomorphism $p_{\boldsymbol{i}}|_C : C \to p(C)$ by p_C.

Cell Decomposition. Let $n \geqslant 1$. A *decomposition of* R^n is a partition of R^n into finitely many cells, obtained by the following recursion:

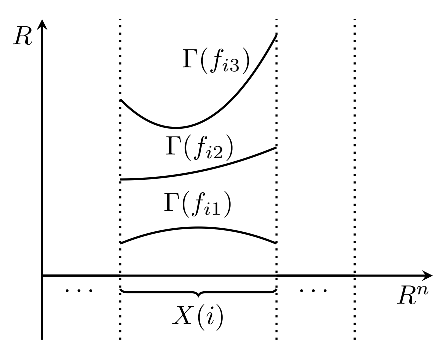
(i) case $n = 1$: points $a_1 < \cdots < a_m$ in R determine a decomposition of $R = R^1$ consisting of $(-\infty, a_1)$, $\{a_1\}$, $(a_1, a_2), \ldots, (a_{m-1}, a_m)$, $\{a_m\}$, (a_m, ∞).

(ii) a decomposition $\mathcal{D}$ of R^{n+1} is a finite partition of R^{n+1} into cells such that $\pi(\mathcal{D}) := \{\pi(C) : C \in \mathcal{D}\}$ is a decomposition of R^n, where $\pi : R^{n+1} \to R^n$ is the projection map given by $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$.

With $X(1), \ldots, X(k)$ the distinct cells of a decomposition $\mathcal{D}$ of R^n, let functions $f_{i1} < \cdots < f_{im_i}$ in $C(X_i)$ be given for $i = 1, \ldots, k$. Then

$$\mathcal{D}_i = \{(-\infty, f_{i1}), \Gamma(f_{i1}), (f_{i1}, f_{i2}), \ldots, \Gamma(f_{im_i}), (f_{im_i}, \infty)\}$$

is a partition of $X(i) \times R$, and $\mathcal{D}^* = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k$ is a decomposition of R^{n+1} with $\mathcal{D} = \pi(\mathcal{D}^*)$. See the figure below. Every decomposition of R^{n+1} is obtained in this manner from a decomposition of R^n.



In these definitions of *cell* and *decomposition* we assumed $n \geqslant 1$, but it is convenient to also consider the one-point set R^0 as the unique cell in R^0, namely as an $\boldsymbol{i}$-cell where $\boldsymbol{i} \in \{0,1\}^0$ is the empty tuple of zeros and ones, and $\{R^0\}$ as the unique decomposition of R^0. In this way clause (i) in these definitions appears as the case $n = 0$ of the corresponding clause (ii). So below we allow $n = 0$.

A decomposition $\mathcal{D}$ of R^n is said to *partition* a set $X \subseteq R^n$ if each cell in $\mathcal{D}$ is either contained in X or disjoint from X (so X is a union of cells in $\mathcal{D}$). We can now state the fundamental Cell Decomposition Theorem:

Theorem 4.2. *For any definable $X_1, \dots, X_m \subseteq R^n$ some decomposition of R^n partitions $X_1, \dots, X_m$. If $X \subseteq R^n$ and $f : X \to R$ are definable, then some decomposition $\mathcal{D}$ of R^n partitions X with continuous $f|_C$ for all cells $C \subseteq X$ in $\mathcal{D}$.*

Some consequences: if the definable set $X \subseteq R^n$ is definably connected, then it is "definably path connected": for any points $p, q \in X$ there is a definable continuous $\gamma : [0,1] \to X$ with $\gamma(0) = p, \gamma(1) = q$. If the underlying ordered field of R is $\mathbb{R}$, then for definable $X \subseteq R^n$, *definably connected* agrees with *connected*.

A *definably connected component* of a definable set $X \subseteq R^n$ is a definably connected definable nonempty subset of X that is maximal with respect to inclusion. (So if $X = \emptyset$, it has no definably connected components.)

Corollary 4.3. *For definable $X \subseteq R^n$, the definably connected components of X are all open and closed in X, and form a finite partition of X.*

Definable families. Let $E \subseteq R^m$ and $X \subseteq E \times R^n \subseteq R^{m+n}$ be definable. For $a \in E$ we set

$$X(a) \ := \ \{x \in R^n : \ (a, x) \in X\}.$$

We view X as describing the family $\big(X(a)\big)_{a \in E}$ of definable subsets of R^n. We call this a *definable family*, and the sections $X(a)$ are the members of the family.

Example. The hypersurfaces in R^2 of degree at most 2 are the members of a semialgebraic family: such a hypersurface is the set of solutions in R^2 of an equation

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \ = \ 0 \quad \text{ with } (a_1, a_2, a_2, a_4, a_5, a_6) \in \mathbb{R}^6 \setminus \{0\},$$

so here $E = R^6 \setminus \{0\}$ and X consists of the points $(a_1, a_2, a_2, a_4, a_5, a_6, x, y) \in E \times R^2$ satisfying the above equation. By the same token, for any n and d the hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant d$ are the members of a semialgebraic family.

Let $\pi : R^{m+n} \to R^m$ be given by $\pi(x_1, \dots, x_{m+n}) = (x_1, \dots, x_m)$.

Proposition 4.4. *Suppose $\mathcal{D}$ is a decomposition of R^{m+n} partitioning X. Then for each $a \in E$,*

$$\mathcal{D}(a) := \{C(a) : C \in \mathcal{D}, a \in \pi(C)\}$$

is a decomposition of R^n partitioning $X(a)$. This gives in particular a finite bound on the number of definably connected components of $X(a)$ independent of $a \in E$.

Dimension. This subsection is taken from [28, Chapter 4, section 1]. It is natural to assign to an $(i_1, \dots, i_n)$-cell C the dimension

$$\dim C \ := \ i_1 + \cdots + i_n \in \{0, \dots, n\},$$

since $p_{(i_1,\dots,i_n)} : R^n \to R^{i_1+\cdots+i_n}$ is definable and maps C homeomorphically onto an open subset of $R^{i_1+\cdots+i_n}$. Such C does not contain any $(j_1, \dots, j_n)$-cell with $j_1 + \cdots + j_n > \dim(C)$. This fact allows us to extend the above dimension to arbitrary nonempty definable $X \subseteq R^n$ by

$$\dim X \ := \ \max\{\dim C : \ C \subseteq X \text{ is a cell}\} \in \mathbb{N}.$$

We also set $\dim \emptyset := -\infty$. Here are some basic facts on dimension:

Proposition 4.5. *Let $X \subseteq R^m$ be definable. Then:*

(i) $\dim X = 0 \iff X$ *is finite and nonempty;*

(ii) $\dim X = m \iff X$ *has nonempty interior in* R^m*;*

(iii) *if $Y \subseteq R^m$ is definable, then* $\dim X \cup Y = \max(\dim X, \dim Y)$*;*

(iv) *if $Y \subseteq R^n$ is definable, then* $\dim X \times Y = \dim X + \dim Y$*;*

(v) *if $f : X \to R^n$ is definable, then* $\dim X \geqslant \dim f(X)$*;*

(vi) *if $f : X \to R^n$ is definable and injective, then* $\dim X = \dim f(X)$*;*

(vii) *if $X \neq \emptyset$, then* $\dim(\mathrm{cl}(X) \setminus X) < \dim X$.

In (v), (vi) we do not assume f is continuous. Here is a stronger version of (v):

Proposition 4.6. *Let $f : X \to R^n$ be definable, $X \subseteq R^m$. For $d \leqslant m$, set*

$$Y(d) \ := \ \{y \in R^n : \dim f^{-1}(y) = d\}.$$

Then $Y(d)$ is definable and $\dim X = \max_{d \leqslant m} d + \dim Y(d)$.

We also have a *local* dimension: Let $X \subseteq R^m$ be definable and $a \in R^m$. Then there is a definable neighborhood V of a in R^m such that $\dim(X \cap U) = \dim(X \cap V)$ for all definable neighborhoods $U \subseteq V$ of a in R^m; thus $\dim(X \cap V)$ is independent of the choice of such V, and we set $\dim_a X := \dim(X \cap V)$ for such V.

Definable Compactness. This subsection and the next are from [28, Chapter 6, section 1]. The ordinary notion of compactness from point set topology is useless in our setting, but we do have a good substitute. Call a set $X \subseteq R^m$ *bounded* if $X \subseteq [-r, r]^m$ for some $r \in R^{>}$.

Proposition 4.7. *If $f : X \to R^n$ is a continuous definable map on a closed and bounded (definable) set $X \subseteq R^m$, then $f(X) \subseteq R^n$ is also closed and bounded.*

This has the expected consequences:

Corollary 4.8. *If $f : X \to R$ is a continuous definable function on a nonempty closed bounded set $X \subseteq R^m$, then f has a maximum and a minimum value on X.*

Corollary 4.9. *If $f : X \to R^n$ is an injective continuous definable map on a closed and bounded set $X \subseteq R^m$, then $f : X \to f(X)$ is a homeomorphism.*

Definable Selection. For any interval (a,b) we can "definably" select a point in it: $(a+b)/2$ if $a,b \in R$; $b-1$ if $a=-\infty$ and $b \in R$; $a+1$ if $a \in R$ and $b=\infty$; 0 if $a=-\infty$ and $b=\infty$. This can be exploited to give two very useful selection principles, the second a consequence of the first:

Proposition 4.10. *Any definable equivalence relation on a definable set $X \subseteq R^n$ has a definable set of representatives, that is, a definable subset of X that has exactly one point in common with each equivalence class. Any definable map $f: X \to R^n$, $X \subseteq R^m$, has a definable right-inverse $g: f(X) \to X$, that is, $f \circ g = \mathrm{id}_{f(X)}$.*

In these last two subsections we got to use the underlying additive group of R, but not yet its multiplication. Accordingly this material goes through in the more general o-minimal setting of [28, Chapter 6] (not needed for our purpose). We now turn to a topic where multiplication does come into play.

The rest of the section is from [28, Chapter 7].

Differentiability. In this subsection we don't need o-minimality or definability, and R can be any ordered field. The elementary facts stated here have the same proofs as for $R = \mathbb{R}$. For $a,b \in R^n$ we set $a \cdot b := a_1b_1 + \cdots + a_nb_n \in R$ (dot product). Let $I \subseteq R$ be open. A map $f: I \to R^n$ is said to be differentiable at a point $a \in I$ with derivative $b \in R^n$ if

$$\lim_{t\to 0} \frac{1}{t}\big(f(a+t) - f(a)\big) \;=\; b.$$

In that case f is continuous at a and we set $f'(a) := b$. If $f,g: I \to R^n$ are differentiable at a, then so are $f+g: I \to R^n$ and $f \cdot g: I \to R = R^1$, with

$$(f+g)'(a) \;=\; f'(a) + g'(a), \quad (f\cdot g)'(a) \;=\; f'(a)\cdot g(a) + f(a)\cdot g'(a),$$

and if in addition $n=1$, g is continuous, and $g(a) \neq 0$, then $f/g: I \setminus g^{-1}(0) \to R$ is differentiable at a with $(f/g)'(a) = \big(f'(a)g(a) - f(a)g'(a)\big)/g(a)^2$. Constant maps $I \to R^n$ are differentiable at every $a \in I$ with derivative $0 \in R^n$, and the inclusion map $I \to R$ is differentiable at every $a \in I$ with derivative $1 \in R$.

Chain Rule: if $f: I \to R$ is continuous, differentiable at $a \in I$, and $f(a) \in J$ with open $J \subseteq R$, and $g: J \to R$ is differentiable at $f(a)$, then $g \circ f: I \cap f^{-1}(J) \to R$ is differentiable at a with $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Next we consider directional derivatives. We consider a map $f: U \to R^n$ with open $U \subseteq R^m$. For a point $a \in U$ and a vector $v \in R^m$ we say that f *is differentiable at a in the v-direction* if the R^n-valued map $t \mapsto f(a+tv)$ (defined on an open neighborhood of $0 \in R$) is differentiable at 0, that is, $\lim_{t\to 0} \frac{1}{t}\big(f(a+tv) - f(a)\big)$ exists in R^n, in which case we set

$$d_a f(v) \;:=\; \lim_{t\to 0} \frac{1}{t}\big(f(a+tv) - f(a)\big) \in R^n.$$

For the standard basis vectors $e_1, \dots, e_m$ of the R-linear space R^m we also write $\frac{\partial f}{\partial x_i}(a)$ for $d_a f(e_i)$.

Let $a \in U$ and let $T: R^m \to R^n$ be an R-linear map. We call f *differentiable at a with differential T* if for every $\varepsilon \in R^>$ we have, for all sufficiently small $v \in R^m$,

$$|f(a+v) - f(a) - T(v)| \;\leqslant\; \varepsilon |v|.$$

Then f is continuous at a and T is uniquely determined by f, a, so we can set $d_a f := T$, a notation consistent with that for directional derivatives: for each vector $v \in R^m$ the map f is differentiable at a in the v-direction with $d_a f(v) = T(v)$ where $d_a f(v)$ denotes the directional derivative defined earlier. For $m = 1$ this notion of differentiability at a agrees with the one defined earlier, with $d_a f(1) = f'(a)$.

The map $f = (f_1, \ldots, f_n) : U \to R^n$ is differentiable at a iff $f_1, \ldots, f_n : U \to R$ are differentiable at a. In that case all partials $\big(\partial f_i / \partial x_j\big)(a)$ exist and the $n \times m$ matrix $\big(\partial f_i / \partial x_j\big)(a)$ is the matrix of $d_a f$ with respect to the standard basis vectors of R^m and R^n. Each R-linear map $R^m \to R^n$ is differentiable at each point of R^m with itself as differential. If the maps $f, g : U \to R^n$ are differentiable at $a \in U$, then $f + g$ and cf for $c \in R$ are differentiable at a with

$$d_a(f+g) \ = \ d_a f + d_a g, \qquad d_a cf \ = \ c \cdot d_a f.$$

Chain Rule: Suppose $U \subseteq R^m$, $V \subseteq R^n$ are open, $a \in U$, $f : U \to R^n$ is continuous, f is differentiable at the point $a \in U$, $f(a) \in V$, and $g : V \to R^l$ is differentiable at $f(a)$. Then $g \circ f : U \cap f^{-1}(V) \to R^l$ is differentiable at a, and

$$d_a(g \circ f) \ = \ \big(d_{f(a)} g\big) \circ d_a f.$$

Preserving Definability and the Mean Value Theorem. We now revert to the setting where R is an o-minimal field and our sets and maps are definable. So let $U \subseteq R^m$ be open and definable, and let $f : U \to R^n$ be definable. Then the set D_f of points (a, v) in $U \times R^m \subseteq R^{2m}$ such that f is differentiable at a in the v-direction is definable, and so is the map $(a, v) \mapsto d_a f(v) : D_f \to R^n$.

Lemma 4.11. *Let $a < b$ in R, and suppose $f : [a, b] \to R$ is definable and continuous, and differentiable at each point of (a, b). Then there is $c \in (a, b)$ with*

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

This "mean value" lemma opens the door to *continuous differentiability.* Let

$$f = (f_1, \ldots, f_n) : U \to R^n$$

be a definable map on a (definable) open set $U \subseteq R^m$. We say that f is *of class* C^1 (or just C^1) if f is differentiable at every point $a \in U$ in the directions $e_1, \ldots, e_m$, and the resulting (definable) functions $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} : U \to R^n$ are continuous. In the next lemma we take matrices with respect to the standard bases of R^m and R^n.

Lemma 4.12. *If f is C^1, then f is differentiable at each point of U and*

$$a \mapsto \textit{matrix of } d_a f \ : \ U \to R^{n \times m}$$

is continuous. Conversely, if f is differentiable at each point of U and the above matrix-valued map is continuous, then f is a C^1-map.

For an R-linear map $T : R^m \to R^n$ we put $|T| := \max\{|Ta| : \ |a| \leqslant 1, \ a \in R^m\}$. (The maximum exists in R since T is definable and continuous.) Thus $|Ta| \leqslant |T| \cdot |a|$ for all $a \in R^m$. Now we can state an extended mean value result:

Lemma 4.13. *Suppose $f : U \to R^n$ is of class C^1, and $a, b \in U$ are such that the line segment $[a, b] := \{(1-t)a + tb : \ 0 \leqslant t \leqslant 1\}$ is contained in U. Then*

$$|f(b) - f(a)| \leqslant |b-a| \cdot \max_{y \in [a,b]} |d_y f|.$$

Smooth Cell Decomposition. It is convenient to extend the notions of C^1-map and C^1-cell. We say that a definable map $f : X \to R^n$, $X \subseteq R^m$, is C^1 if there are a definable open $U \subseteq R^m$ such that $X \subseteq U$, and a definable C^1-map $F : U \to R^n$ such that $f = F|_X$. We define *C^1-cells* as in the recursive definition for *cells*, except that we require the (definable) functions f and g there, when R-valued, to be C^1 instead of just being continuous.

Every inclusion map $X \to R^m$ for definable $X \subseteq R^m$ is C^1. If the definable map $f : X \to R^n$ with $X \subseteq R^m$ is C^1 and the definable map $g : Y \to R^l$ with $Y \subseteq R^n$ is C^1, then $g \circ f : f^{-1}(Y) \to R^l$ is C^1. For definable $f = (f_1, \ldots, f_n) : X \to R^n$ with $X \subseteq R^m$, f is C^1 iff $f_1, \ldots, f_n$ are C^1.

The following is a C^1-version of cell decomposition.

Theorem 4.14. *For any definable $X_1, \ldots, X_m \subseteq R^n$ there is a decomposition of R^n into C^1-cells partitioning $X_1, \ldots, X_m$. If $X \subseteq R^n$ and $f : X \to R$ are definable, then there is a decomposition of R^n into C^1-cells which partitions X such that $f|_C$ is C^1 for each cell $C \subseteq X$ of the decomposition.*

C^k-maps. Let $k \geqslant 1$ below. Let $f = (f_1, \ldots, f_n) : U \to R^n$ be definable, with (definable) open $U \subseteq R^m$. By recursion on k we specify what it means for f to be a C^k-map. For $k = 1$ this has been defined earlier, and for $k > 1$ we say that f is C^k if f is C^1 and every partial $\partial f_i / \partial x_j : U \to R$ $(1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m)$ is C^{k-1}.

We also extend being C^k to definable maps $f : X \to R^n$ with $X \subseteq R^m$ not necessarily open, just as we did for $k = 1$, and likewise we define the notion of a C^k-cell just like we did for $k = 1$. The remarks we made above about this extended notion of definable C^1-map go through when "C^1" is replaced by "C^k" everywhere. The C^1-cell decomposition theorem goes through with "C^1" replaced by "C^k" everywhere.

Semialgebraic Functions. We finish this section with some facts on real semialgebraic functions, and use this to determine the algebraic part of the set

$$X \ = \ \{(a, b, c) \in \mathbb{R}^3 : \ 1 < a, b < 2, \ c = a^b\} \ \subseteq \ \mathbb{R}^3$$

of the Example in Section 5.1.

Let $f : I \to \mathbb{R}$ be a continuous function on an interval $I \subseteq \mathbb{R}$. Then f is semialgebraic (that is, its graph as a subset of $\mathbb{R}^2$ is semialgebraic) iff for some nonzero polynomial $P \in \mathbb{R}[x, y]$ we have $P\big(t, f(t)\big) = 0$ for all $t \in I$. (See [28, Chapter 2] for a detailed treatment of real semialgebraic sets and functions, in particular Exercise 3 in (3.7) there, with Hint on p. 169.) Thus if f is real analytic, and its restriction to some subinterval of I is semialgebraic, then f is semialgebraic.

Suppose f is semialgebraic and $I = (0, b)$ with $b \in (0, \infty]$. Then either $f(t) = 0$ for all sufficiently small $t > 0$, or for some $q \in \mathbb{Q}$ and $c \in \mathbb{R}^{\times}$ we have $f(t)/t^q \to c$ as $t \downarrow 0$. (See for example the end of [26].) Combining these facts we see that the following real analytic functions on $(0, \infty)$ cannot be semialgebraic on any subinterval of $(0, \infty)$: for $r \in \mathbb{R} \setminus \mathbb{Q}$ the function $t \mapsto t^r$; for $a \in (1, \infty)$ the function $t \mapsto a^t$; the function $t \mapsto \log t$.

By semialgebraic cell decomposition the algebraic part Y^{alg} of any set $Y \subseteq \mathbb{R}^n$ is the union of the sets $\mathrm{cl}(C) \cap Y$ with $C \subseteq Y$ a 1-dimensional semialgebraic cell. We can now prove the fact stated in the Introduction for the above $X \subseteq \mathbb{R}^3$ that

$$X^{\mathrm{alg}} \;:=\; \bigcup_{q \in \mathbb{Q} \cap (1,2)} X_q.$$

The inclusion $\supseteq$ is clear. The sets X_q ($q \in \mathbb{Q} \cap (1,2)$) are closed in X, so given any 1-dimensional semialgebraic cell $C \subseteq X$ it suffices to show that $C \subseteq X_q$ for some $q \in \mathbb{Q} \cap (1,2)$. Now C is a $(0,0,1)$-cell, or a $(0,1,0)$-cell, or a $(1,0,0)$-cell. But X does not contain any $(0,0,1)$-cell. Also C cannot be a $(0,1,0)$-cell: if it were, then for a fixed $a \in (1,2)$ and an interval $I \subseteq (1,2)$ the function $t \mapsto a^t$ on I would be semialgebraic, which is false. Finally, suppose C is a $(1,0,0)$-cell. Then $C = \{\big(t, f(t), t^{f(t)}\big) : t \in I\}$ where $f : I \to \mathbb{R}$ is a continuous semialgebraic function on an interval $I \subseteq (1,2)$ (and $t \mapsto t^{f(t)} : I \to \mathbb{R}$ is also semialgebraic). Consider any interval $J \subseteq I$ on which f is of class C^1. Then taking the logarithmic derivative of $t \mapsto t^{f(t)} = \mathrm{e}^{f(t)\log t}$ on J gives that $f'(t)\log t + f(t)/t$ is semialgebraic as a function of $t \in J$, and so $f' = 0$ on J. Using several such J we see that f is constant on I. This constant value must be a rational $q \in (1,2)$, so $C \subseteq X_q$.

4.3 Some model theory

In Section 7.4 we use the notion of an $\aleph_0$-saturated elementary extension, requiring a little excursion into model theory. We shall give precise definitions of the necessary model-theoretic notions, motivating them by examples, and stating carefully a few needed results. For most proofs we refer to [2, Appendix B]; there the basics of model theory are developed in the setting of many-sorted structures, while here we stay with one-sorted structures (which have only one underlying set, while a many-sorted structure has a *family* of underlying sets). We end this section with a detailed explanation of the deployment of saturation in Chapter 7.

A *language* is a set L whose elements are called symbols, each symbol $s \in L$ being equipped with a natural number $\mathrm{arity}(s) \in \mathbb{N}$. These symbols are either *relation symbols* or *function symbols*, and L is the disjoint union of L^{r}, its subset of relation symbols, and L^{f}, its subset of function symbols.

Below L is a language. Let $\mathcal{M}$ be an *L-structure*, that is, a triple

$$\mathcal{M} \;=\; \big(M; (R^{\mathcal{M}})_{R \in L^{\mathrm{r}}}, (F^{\mathcal{M}})_{F \in L^{\mathrm{f}}}\big)$$

consisting of a *nonempty* set M, an m-ary relation $R^{\mathcal{M}} \subseteq M^m$ on M for each $R \in L^{\mathrm{r}}$ of arity m, and an n-ary function $F^{\mathcal{M}} : M^n \to M$ on M for each $F \in L^{\mathrm{f}}$ of arity n. We call M the *underlying set* of $\mathcal{M}$, we think of a symbol $R \in L^{\mathrm{r}}$ as *naming* the corresponding relation $R^{\mathcal{M}}$ on M, and likewise for $F \in L^{\mathrm{f}}$.

Thus a nullary function symbol (also called a *constant symbol*) names a function $M^0 \to M$, to be identified with its unique value in M, so a constant symbol names a distinguished element of M. Usually we drop the superscripts $\mathcal{M}$ in $R^{\mathcal{M}}$ and $F^{\mathcal{M}}$ when $\mathcal{M}$ is understood from the context, the distinction between the symbols and what they name to be kept in mind. We shall also feel free to denote $\mathcal{M}$ and its underlying set M by the same letter, when convenient. The reason we need the distinction between symbols and what they name in a particular L-structure is that we have to be able to say that a statement expressed in terms of these symbols holds in, say, two *different* L-structures $\mathcal{M}$ and $\mathcal{N}$.

We need to consider two (unrelated) ways of increasing $\mathcal{M}$. The *first* is when L is a sublanguage of L'. Then an *L'-expansion of* $\mathcal{M}$ is an L'-structure $\mathcal{M}'$ with the same underlying set as $\mathcal{M}$ and where the symbols of L name the same relations and functions in $\mathcal{M}$ as in $\mathcal{M}'$. We also say that then $\mathcal{M}'$ *expands* $\mathcal{M}$.

Example. The language L_{OF} of ordered fields has a binary relation symbol $<$, constant symbols 0 and 1, a unary function symbol $-$, and binary function symbols $+$ and $\cdot$. Any ordered field K is viewed as an L_{OF}-structure by having $<$ name the (strict) ordering of the field, and the function symbols name the functions on K that these symbols traditionally denote. Thus an ordered field K *expands* its underlying field. Equipping the ordered field $\mathbb{R}$ of real numbers with the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ gives the expansion $\mathbb{R}_{\exp}$ of $\mathbb{R}$. This is not exactly how we specified $\mathbb{R}_{\exp}$ in Section 4.2, but the difference is immaterial: the two descriptions lead to the same sets $X \subseteq \mathbb{R}^n$ being definable in $\mathbb{R}_{\exp}$, see below. For *model-theoretic* use we take $\mathbb{R}_{\exp}$ as the above expansion of $\mathbb{R}$.

A *second* way: $\mathcal{M}$ *is a substructure of* $\mathcal{N}$ (or $\mathcal{N}$ *is an extension of* $\mathcal{M}$); notation: $\mathcal{M} \subseteq \mathcal{N}$. This means: $\mathcal{N} = (N; \dots)$ is an L-structure, $M \subseteq N$, $R^{\mathcal{N}} \cap M^m = R^{\mathcal{M}}$ for m-ary $R \in L^{\mathrm{r}}$, and $F^{\mathcal{M}}(a) = F^{\mathcal{N}}(a)$ for n-ary $F \in L^{\mathrm{f}}$ and $a \in M^n$. For example, if K_1 and K_2 are ordered fields viewed as L_{OF}-structures, $K_1 \subseteq K_2$ means that K_1 is an ordered subfield of K_2 (so the ordering of K_2 restricts to the ordering of K_1).

The 0-definable (or absolutely definable) sets of $\mathcal{M}$ are the sets $X \subseteq M^n$ for $n = 0, 1, 2, \dots$ obtained recursively as follows:

(D1) $R^{\mathcal{M}} \subseteq M^m$ for m-ary $R \in L^{\mathrm{r}}$ and $\mathrm{graph}(F^{\mathcal{M}}) \subseteq M^{n+1}$ for n-ary $F \in L^{\mathrm{f}}$ are 0-definable.

(D2) if $X, Y \subseteq M^n$ are 0-definable, then so are $X \cup Y$ and $M^n \setminus X$;

(D3) if $X \subseteq M^n$ is 0-definable, then so are $X \times M, M \times X \subseteq M^{n+1}$;

(D4) for any $i < j$ in $\{1, \dots, n\}$ the diagonal $\{(a_1, \dots, a_n) : \ a_i = a_j\} \subseteq M^n$ is 0-definable;

(D5) if $X \subseteq M^{n+1}$ is 0-definable, then so is $\pi(X) \subseteq M^n$, where $\pi : M^{n+1} \to M^n$ is given by $\pi(a_1, \dots, a_n, a_{n+1}) = (a_1, \dots, a_n)$.

Thus the 0-definable sets $X \subseteq M^n$ are exactly those that belong to the smallest structure on M that contains all sets $R^{\mathcal{M}}$ with $R \in L^{\mathrm{r}}$ and all sets $\mathrm{graph}(F^{\mathcal{M}})$ with $F \in L^{\mathrm{f}}$. If $X \subseteq M^n$ is a 0-definable set of $\mathcal{M}$ and the ambient $\mathcal{M}$ is clear from the context, we also just say that X is 0-definable. A map $f : X \to M^n$ with $X \subseteq M^m$ is said to be 0-definable (in $\mathcal{M}$) if its graph as a subset of M^{m+n} is 0-definable in $\mathcal{M}$. In that case its domain X is 0-definable and for every 0-definable $X' \subseteq X$ its image $f(X') \subseteq M^n$ is 0-definable.

We need a notation system to describe 0-definable sets in a uniform way in varying L-structures. Towards this we assume that in addition to the symbols of L we have an infinite set Var of symbols, called *variables* (with Var disjoint from the language L and independent of L). Given a tuple $x = (x_1, \dots x_m)$ of distinct variables we define L-terms t in x recursively as follows: each x_i with $1 \leqslant i \leqslant m$ is an L-term in x, and for n-ary $F \in L^{\mathrm{f}}$ and L-terms $t_1, \dots, t_n$ in x, the expression $F(t_1, \dots, t_n)$ is an L-term in x. Formally, these expressions are words on some alphabet together with the specification of the tuple x, but we prefer not to go into detail on such syntactical matters. We let $t(x)$ indicate a term t in x. An L-term $t = t(x)$ names in an obvious way a function $t^{\mathcal{M}} : M^m \to M$, with x_i naming the function $(a_1, \dots, a_m) \mapsto a_i : M^m \to M$. Functions named by L-terms are 0-definable in $\mathcal{M}$.

For example, when K is an ordered field, any L-term t in $x = (x_1, \dots, x_m)$ names a function $K^m \to K$ given by a polynomial in $\mathbb{Z}[x_1, \dots, x_m]$, and each such polynomial function is named by an L-term in x. (Different terms can name the same function: $x + (-x)$ and 0 are different as terms in the single variable x, but name the same function $K \to K$, which takes the constant value 0.)

Going back to our L-structure $\mathcal{M}$ we introduce for every element $a \in M$ a constant symbol $\underline{a} \notin L$ as a name for a, with $\underline{a} \neq \underline{b}$ for all $a \neq b$ in M. For every set $A \subseteq M$ we extend L to the language $L_A := L \cup \{\underline{a} : \ a \in A\}$, and expand $\mathcal{M}$ to the L_A-structure $\mathcal{M}_A$ with $\underline{a}$ naming a for $a \in A$. A set $X \subseteq M^n$ is said to be A-definable (or definable over A) in $\mathcal{M}$ if X is 0-definable in $\mathcal{M}_A$. When $A = M$ we just use write "definable" instead of "M-definable". A set $X \subseteq M^n$ is A-definable (in $\mathcal{M}$) iff $X = Y(a)$ for some 0-definable set $Y \subseteq M^{m+n}$ and some $a \in A^m$. (Here for $Y \subseteq M^{m+n}$ and $a \in M^m$ we set $Y(a) := \{b \in M^n : (a, b) \in Y\}$.)

Examples. For an algebraically closed field $\boldsymbol{k}$, viewed as an L-structure with $L = \{0, 1, -, +, \cdot\}$ (the language of rings), the subsets of $\boldsymbol{k}^n$ definable in $\boldsymbol{k}$ are exactly the so-called constructible subsets of $\boldsymbol{k}^n$: the finite unions of sets $X \setminus Y$ with X, Y Zariski-closed subsets of $\boldsymbol{k}^n$. (This is basically the constructibility theorem of Tarski-Chevalley: the image of a constructible subset of $\boldsymbol{k}^{n+1}$ in $\boldsymbol{k}^n$ under the projection map $(a_1, \ldots, a_{n+1}) \mapsto (a_1, \ldots, a_n) : \boldsymbol{k}^{n+1} \to \boldsymbol{k}^n$ is constructible in $\boldsymbol{k}^n$.) The same holds with "A-definable" and "A-constructible" instead of "definable" and "constructible" for any set $A \subseteq \boldsymbol{k}$, where an A-constructible subset of $\boldsymbol{k}^n$ is a finite union of sets $X \setminus Y$ with $X, Y \subseteq \boldsymbol{k}^n$ given by the vanishing of polynomials in $D[x_1, \ldots, x_n]$ where D is the subring of $\boldsymbol{k}$ generated by A.

For us the more relevant example is when K is an ordered real closed field. Then the subsets of K^n definable in K are exactly the semialgebraic subsets of K^n, that is, the finite unions of sets (with $f, g_1, \ldots, g_m$ ranging over $K[x_1, \ldots, x_n]$)

$$\{a \in K^n : \ f(a) = 0,\ g_1(a) > 0, \ldots, g_m(a) > 0\}.$$

This is the Tarski-Seidenberg theorem (like the Tarski-Chevalley theorem, but with "semialgebraic" instead of "constructible"). Requiring the polynomials $f, g_1, \ldots, g_m$ above to have coefficients in $\mathbb{Z}$, we obtain likewise exactly the subsets of K^n that are 0-definable in K.

Saturation. This notion functions as a kind of compactness for definable sets. Let κ be a cardinal. An L-structure $\mathcal{M}$ is said to be *κ-saturated* if for every set $A \subseteq M$ of cardinality $< \kappa$ and any family (X_i) of A-definable subsets of M with the finite intersection property we have $\bigcap_i X_i \neq \emptyset$. (The *finite intersection property for* (X_i) says that $X_{i_1} \cap \cdots \cap X_{i_n} \neq \emptyset$ for all indices $i_1, \ldots, i_n$.) This property of families of definable subsets of M is inherited by families of definable subsets of M^m for any m. One can indeed think of this in terms of compactness: if $\mathcal{M}$ is κ-saturated, then for $A \subseteq M$ of cardinality $< \kappa$, the A-definable subsets of M^m are a basis for a topology on M^m, the A-topology, which makes M^m a compact hausdorff space in which these A-definable sets are exactly the open-and-closed sets. We need this only for $\kappa = \aleph_0$, which means that in the definition above we restrict to *finite* $A \subseteq M$. For $\kappa = \aleph_1$ the restriction is to *countable* $A \subseteq M$.

For example, any algebraically closed field of infinite transcendence degree over its prime field is $\aleph_0$-saturated, and the field $\mathbb{C}$ of complex numbers is even $\aleph_1$-saturated. The ordered field $\mathbb{R}$ is not even 1-saturated, since $\bigcap_n (n, \infty) = \emptyset$. [The referee asked for an explicit example of an $\aleph_0$-saturated elementary extension of the ordered field of real numbers. An attractive example is the real closed ordered field of *surreal numbers of countable length*, which is actually $\aleph_1$-saturated; for surreal numbers, see Gonshor [38]. In this connection we mention that a real closed ordered field R is $\aleph_1$-saturated iff for all countable subsets $A < B$ of R (that is $a < b$ for all $a \in A$ and $b \in B$), there is a $c \in R$ such that $A < c < B$.] Any finite structure (a structure with finite underlying set) is κ-saturated for every κ.

Towards the study of a structure $\mathcal{M}$ of interest we can always pass to an $\aleph_1$-saturated extension $\mathcal{N}$ with the same elementary properties, do our work in $\mathcal{N}$ and then pass the information gained back to $\mathcal{M}$. This will be made precise below. To make sense of "the same elementary properties" we need a notation system for definable sets. This is the reason for introducing formulas and sentences below.

Formulas and sentences. Let $y = (y_1, \dots, y_n)$ be a tuple of distinct variables. We define L-formulas ϕ in y inductively as follows:

(i) The *atomic* L-formulas in y are the expressions

$$\top, \quad \bot, \quad R\big(t_1, \dots, t_m\big), \quad \text{and } t_1 = t_2$$

for m-ary $R \in L^{\mathrm{r}}$ and L-terms $t_1, \dots, t_m$ in y, and L-terms t_1, t_2 in y.

(ii) Given any L-formulas ϕ and ψ in y, we have new L-formulas in y:

$$\neg\phi, \quad \phi \vee \psi, \quad \text{and } \phi \wedge \psi.$$

(iii) Let ϕ be a formula in $(y_1, \dots, y_i, z, y_{i+1}, \dots, y_n)$, where $0 \leqslant i \leqslant n$ and z is a variable different from $y_1, \dots, y_n$; then

$$\exists z\phi \quad \text{and} \quad \forall z\phi$$

are new L-formulas in y.

Formally, these formulas in y are words on a certain alphabet, together with the specification of the tuple y. We also write $\phi(y)$ to indicate that we are dealing with a formula ϕ in y. Each L-formula $\phi = \phi(y)$ names (we also say: *defines*) a 0-definable set $\phi^{\mathcal{M}} \subseteq M^n$: the atomic formulas $\top$ and $\bot$ name the subsets M^n and $\emptyset$ of M^n, and the atomic formulas $R(t_1, \dots, t_m)$ and $t_1 = t_2$ as above name the sets

$$\{a \in M^n : \ (t_1^{\mathcal{M}}(a), \dots, t_m^{\mathcal{M}}(a)) \in R^{\mathcal{M}}\} \text{ and } \{a \in M^n : \ t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a)\},$$

and for L-formulas ϕ, ψ as above,

$$(\neg\phi)^{\mathcal{M}} = M^n \setminus \phi^{\mathcal{M}}, \quad (\phi \vee \psi)^{\mathcal{M}} \ := \ \phi^{\mathcal{M}} \cup \psi^{\mathcal{M}}, \ (\phi \wedge \psi)^{\mathcal{M}} \ = \ \phi^{\mathcal{M}} \cap \psi^{\mathcal{M}},$$

and for an L-formula ϕ in $(y_1, \dots, y_i, z, y_{i+1}, \dots, y_n)$ as above: $(\exists z\phi)^{\mathcal{M}}$ is the set

$$\{(a_1, \dots, a_n) \in M^n : \ (a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \in \phi^{\mathcal{M}} \text{ for some } b \in M\},$$

that is, the image of $\phi^{\mathcal{M}} \subseteq M^{n+1}$ under the projection map

$$(a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \mapsto (a_1, \dots, a_n) : M^{n+1} \to M^n,$$

and $(\forall z\phi)^{\mathcal{M}} := (\neg\exists z\neg\phi)^{\mathcal{M}}$, which equals the set

$$\{(a_1, \dots, a_n) \in M^n : \ (a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \in \phi^{\mathcal{M}} \text{ for every } b \in M\}.$$

The 0-definable subsets of M^n are exactly the sets $\phi^{\mathcal{M}}$ with ϕ an L-formula in y. Likewise for $A \subseteq M$, the A-definable subsets of M^n are exactly the sets $\phi^{\mathcal{M}_A}$ with ϕ an L_A-formula in y, but for convenience we write this also as $\phi^{\mathcal{M}}$.

The L-formulas ϕ in $y = (y_1, \dots, y_n)$ with $n = 0$ have a special status and are called L-*sentences*, typically denoted by σ. One can think of a sentence as making an assertion. Formally, a sentence σ names a set

$\sigma^{\mathcal{M}} \subseteq M^0$, so it equals M^0, in which case we say that σ is *true in* $\mathcal{M}$ (notation: $\mathcal{M} \models \sigma$), or it is empty, in which case we way that σ is *false in* $\mathcal{M}$. For example, if L is the language of rings and x, y are distinct variables, then the L-sentence $\forall x \exists y (x = y \cdot y)$ is true in exactly those fields in which every element is a square.

Elementary extensions. An *elementary extension* of the L-structure $\mathcal{M}$ is an extension $\mathcal{N} \supseteq \mathcal{M}$ of $\mathcal{M}$ such that the same L_M-sentences are true in $\mathcal{M}$ and $\mathcal{N}$ (where of course for $a \in M$ the constant symbol $\underline{a}$ names a in both $\mathcal{M}$ and $\mathcal{N}$). Notation: $\mathcal{M} \preccurlyeq \mathcal{N}$. Here are two wellknown situations where this is the case: any algebraically closed field is an elementary extension of any algebraically closed subfield, any real closed field is an elementary extension of any real closed subfield.

Suppose $\mathcal{M} \preccurlyeq \mathcal{N}$. Then for any L_M-formula $\phi = \phi(x_1, \dots, x_n)$ we have $\phi^{\mathcal{M}} = \phi^{\mathcal{N}} \cap M^n$. Moreover, if $\psi = \psi(x_1, \dots, x_n)$ is a second L_M-formula and $\phi^{\mathcal{M}} = \psi^{\mathcal{M}}$, then $\phi^{\mathcal{N}} = \psi^{\mathcal{N}}$, so a definable set $X = \phi^{\mathcal{M}} \subseteq M^n$ (of $\mathcal{M}$) yields a definable set $X(\mathcal{N}) = \phi^{\mathcal{N}} \subseteq N^n$ that does not depend on the choice of defining formula ϕ.

To profit from saturation and elementary extensions we use:

Proposition 4.15. *Any L-structure has an $\aleph_1$-saturated elementary extension.*

Indeed, for any L-structure $\mathcal{M}$ and nonprincipal ultrafilter U on the set $\mathbb{N}$, the ultrapower $\mathcal{M}^{\mathbb{N}}/U$ is an $\aleph_1$-saturated elementary extension of $\mathcal{M}$, where $\mathcal{M}$ is identified with a substructure of $\mathcal{M}^{\mathbb{N}}/U$ via the diagonal embedding; this is a remark intended for those who know about ultrapowers. In Section 7.4 we only need $\aleph_0$-saturation instead of the stronger $\aleph_1$-saturation.

Revisiting o-minimality. Let K be an *expansion* of an ordered field. Among the definable subsets of K in this expansion are at least the open intervals $(a, b)_K$ with $a < b$ in $K \cup \{-\infty, +\infty\}$, and thus the finite unions of such open intervals and singletons $\{a\}$ with $a \in K$. We say that K is *o-minimal* if there are no other subsets of K definable in this expansion. To see how this is related to the concept of o-minimality considered in Section 4.2, let $\mathrm{Def}^n(K)$ be the collection of sets $X \subseteq K^n$ that are definable in this expansion K. Note that then K is o-minimal if and only if the family $\big(\mathrm{Def}^n(K)\big)$ is an o-minimal structure on the underlying ordered field of K. In particular, if K is o-minimal, then the underlying ordered field of K is real closed.

Lemma 4.16. *If K is o-minimal, then so is any elementary extension of K.*

Proof. Assume K is o-minimal, and $K \preccurlyeq K^*$, so the underlying ordered field of K^* extends the underlying ordered field of K. Let $X \subseteq K^*$ be definable. Then we have a definable set $Y \subseteq K^{n+1}$ and a point $b \in (K^*)^n$ such that $X = Y^{K^*}(b)$. Take $N \in \mathbb{N}$ and cells $C_1, \dots, C_N$ in K^{n+1} such that $Y = C_1 \cup \dots \cup C_N$. Then for every $a \in K^n$ the set $Y(a)$ is a union of at most N sets $\{c\}$ with $c \in K$, and intervals of K. With K an L-structure, this fact can be expressed by a certain L_K-sentence being true in K, hence in K^*, which then means in particular that $X = Y^{K^*}(b)$ is a union of at most N sets $\{c\}$ with $c \in K^*$, and intervals of K^*. $\square$

Let K be o-minimal and $K \preccurlyeq K^*$. To each definable set $X \subseteq K^n$ we associate the set $X^* := X(K^*) \subseteq (K^*)^n$, which is definable (over K) in K^*. If $C \subseteq K^n$ is an $(i_1, \dots, i_n)$-cell in the sense of K, then $C^* \subseteq (K^*)^n$ is an $(i_1, \dots, i_n)$-cell in the sense of K^*. It follows that for definable $X \subseteq K^n$ we have $\dim X = \dim X^*$ where the dimension on the left is in the sense of K, and the dimension on the right is in the sense of K^*.

In Section 4.2 and the rest of the chapters in this part of the thesis, we use the letter R to refer to an o-minimal field, but in this section we used R to indicate a relation symbol. That is why in this section we prefer the letter K when dealing with o-minimal expansions of ordered fields and o-minimal fields. The distinction between the two concepts (*o-minimal expansion of an ordered field* and *o-minimal field*) is often immaterial: we saw that an o-minimal expansion K of an ordered field gives rise to an o-minimal field with the same underlying ordered field and the same definable sets.

When considering *elementary extensions* and $\aleph_0$*-saturation*, the distinction is significant: When referring to an *elementary extension of an o-minimal field* we really mean an elementary extension of an o-minimal expansion K of an ordered field, so K is then an L-structure for a suitable language $L \supseteq L_{\mathrm{OF}}$. Likewise when referring to an o-minimal field as being $\aleph_0$-saturated, we mean: an L-structure K that gives rise to this o-minimal field is $\aleph_0$-saturated.

How to use the above? This subsection is designed to help the reader understand the precise use of saturation in Chapter 7. We first show how Theorem 7.1 follows from it being true when the ambient o-minimal field is $\aleph_0$-saturated. So let K be an o-minimal field, $X \subseteq K^m$ a strongly bounded definable set, and $k \geqslant 1$; we need to show that X has a k-parametrization. We can assume $X \neq \emptyset$, and set $l := \dim X$. Take $N \in \mathbb{N}$ such that $X \subseteq [-N, N]^m_K$. As explained, K is viewed as an L-structure for a language $L \supseteq L_{\mathrm{OF}}$. Fix an L_K-formula $\phi(x)$, $x = (x_1, \dots, x_m)$, such that $X = \phi^K$. Take an $\aleph_0$-saturated elementary extension K^* of K. Then K^* is an o-minimal field and

$$X^* \ = \phi^{K^*} \ \subseteq \ [-N, N]^m_{K^*}$$

is strongly bounded, and so has a k-parametrization $\{f_1, \dots, f_M\}$ (with respect to K^*), since Theorem 7.1 was established in Section 7.4 when the ambient o-minimal field is $\aleph_0$-saturated. For $\mu = 1, \dots, M$ we have $f_\mu : (0,1)^l_{K^*} \to (K^*)^m$, so the graph of f_μ is defined in K^* by an L_{K^*}-formula $\phi_\mu(b, v, x)$ with $\phi_\mu = \phi_\mu(u, v, x)$ an L_K-formula, $u = (u_1, \dots, u_p)$, $v = (v_1, \dots, v_l)$ and $b \in (K^*)^p$ (the same p and b for all μ, without loss of generality). The fact that there exists $b \in (K^*)^p$ such that $\phi_1(b, v, x), \dots, \phi_M(b, v, x)$ define in K^* the graphs of functions of a k-parametrization of X^* can be expressed by a certain L_K-sentence $\exists u \theta(u)$ being true in K^*. (This sentence is complicated but its construction is routine and just mimicks the definitions of the various notions involved.) Hence this sentence $\exists u \theta(u)$ is also true in K, which then means that for some $a \in K^m$ the formulas $\phi_1(a, v, x), \dots, \phi_M(a, v, x)$ define in K the graphs of functions of a k-parametrization of X.

In a very similar way Theorem 7.2 follows from the fact, established in Section 7.4, that it is true when the ambient o-minimal field is $\aleph_0$-saturated.

Next we explain the use of "$\aleph_0$-saturation plus Definable Selection" in obtaining Corollary 7.17 as a consequence of Corollary 7.16. (The other use of this device earlier in Section 7.4 is along the same lines. The argument we give may seem lengthy, but such arguments are utterly routine in model theory and are therefore usually not spelled out but left to the reader.)

We are now dealing with an o-minimal field K which is $\aleph_0$-saturated when viewed as an L-structure for suitable $L \supseteq L_{\mathrm{OF}}$ as before. We are given d, k, m, n with $k, n \geqslant 1$ and definable $E \subseteq K^m$ and definable $Z \subseteq E \times [-1, 1]^n \subseteq K^{m+n}$ with $\dim Z(s) = d$ for all $s \in E$. Take a finite $A \subseteq K$ such that E and Z are A-definable. Corollary 7.16 yields for every $s \in E$ a definable C^k-map

$$f = (f_1, \dots, f_N) : (0,1)^d \to (K^n)^N = K^{Nn}$$

with $N = N(s) \in \mathbb{N}$ depending on s, such that (i) and (ii) of that corollary hold for $\Phi := \{f_1, \dots, f_N\}$ and

$X(s)$ in the role of X. For each L-formula $\phi = \phi(u, x, y)$,

$$u = (u_1, \ldots, u_M), \quad x = (x_1, \ldots, x_d), \quad y = (y_{11}, \ldots, y_{Nn})$$

with $M, N \in \mathbb{N}$ depending on ϕ we consider the A-definable set $E_\phi \subseteq K^m$ of all $s \in E$ such that for some $b \in K^M$ the L_K-formula $\phi(b, x, y)$ defines the graph of a map f parametrizing $X(s)$ as above. Since K is $\aleph_0$-saturated and E is covered by the sets E_ϕ, it is covered by finitely many of them, say by $E_{\phi_1}, \ldots, E_{\phi_e}$,

$$\phi_i \;=\; \phi_i(u_1, \ldots, u_{M(i)}, x, y_{11}, \ldots, y_{N(i)n}) \qquad (i = 1, \ldots, e).$$

For $i = 1, \ldots, e$, let $E(i)$ be the definable set of all $s \in E$ with $s \in E_{\phi_i}$ and $s \notin E_{\phi_j}$ for $1 \leqslant j < i$. Definable selection then yields for such i a definable map

$$s \mapsto b_i(s) : E(i) \to K^{M(i)}$$

such that for every $s \in E(i)$ the L_K-formula $\phi_i(b_i(s), x, y)$ defines in K the graph of a C^k-map $f : (0,1)^d \to (K^n)^{N(i)}$ parametrizing $X(s)$ as specified earlier. Now E is the disjoint union of $E(1), \ldots, E(e)$. Adding suitable constant maps it is routine to obtain from this for $N = \max_i N(i)$ a definable set $F \subseteq E \times K^d \times K^{Nn}$ for which the conclusion of Corollary 7.17 holds.

CHAPTER 5

The Pila-Wilkie Counting Theorem

5.1 The statement

First some notation needed to state the theorem. We define the *multiplicative height function* $\mathrm{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathrm{H}(\frac{a}{b}) \ := \ \max(|a|,\ |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Thus $\mathrm{H}(0) = \mathrm{H}(1) = \mathrm{H}(-1) = 1$, and for $q \in \mathbb{Q}$ we have $\mathrm{H}(q) \geqslant 2$ if $q \notin \{0, 1, -1\}$, $\mathrm{H}(q) = \mathrm{H}(-q)$, and $\mathrm{H}(q^{-1}) = \mathrm{H}(q)$ for $q \neq 0$. For $a = (a_1, ..., a_n) \in \mathbb{Q}^n$,

$$\mathrm{H}(a) \ := \ \max\{\mathrm{H}(a_1), \ldots, \mathrm{H}(a_n)\} \in \mathbb{N}.$$

Let $X \subseteq \mathbb{R}^n$. We set $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$. Throughout T ranges over real numbers $\geqslant 1$, and we set $X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) :\ \mathrm{H}(a) \leqslant T\}$ be the (finite) set of rational points of X of height $\leqslant T$, and set $\mathrm{N}(X, T) := \#X(\mathbb{Q}, T) \in \mathbb{N}$. The *algebraic part* of X, denoted by X^{alg}, is the union of the connected infinite semialgebraic subsets of X. So for $n \geqslant 1$, the interior of X (in $\mathbb{R}^n$) is part of X^{alg}.

Example. The set $X := \{(a, b, c) \in \mathbb{R}^3 :\ 1 < a, b < 2,\ c = a^b\}$ is definable in the o-minimal field $\mathbb{R}_{\exp}$. (See the subsection "O-Minimal Structures" in Section 4.2 for $\mathbb{R}_{\exp}$.) For rational $q \in (1, 2)$, we have a semialgebraic curve

$$X_q \ := \ \{(a, q, c) :\ c = a^q\} \ \subseteq \ X,$$

and X^{alg} is the union of those X_q (proved at the end of Section 4.2).

We also set

$$X^{\mathrm{tr}} \ := \ X \setminus X^{\mathrm{alg}} \qquad (\textit{the transcendental part of } X).$$

We can now state the Pila-Wilkie theorem, also called the *Counting Theorem*:

Theorem 5.1. *Let $X \subseteq \mathbb{R}^n$ with $n \geqslant 1$ be definable in some o-minimal expansion of $\mathbb{R}$. Then for all ε there is a c such that for all T,*

$$\mathrm{N}(X^{\mathrm{tr}}, T) \ \leqslant \ cT^{\varepsilon}.$$

Roughly speaking, it says there are few rational points on the transcendental part of a set definable in an o-minimal expansion of $\mathbb{R}$: the number of such points grows slower than any power T^{ε} with T bounding their height. To apply the counting theorem one needs to describe X^{alg} in some useful way. This typically involves *Ax-Schanuel* type transcendence results.

Note that $X^{\mathrm{tr}}(\mathbb{Q}) = \emptyset$ in the example above, so the theorem is trivial for this X. We shall include a refinement, Theorem 5.8, which is nontrivial for this X.

The proof of Theorem 5.1 depends on two intermediate results. The first of these has nothing to do with o-minimality. To state it we define for $k, n \geqslant 1$ and $X \subseteq \mathbb{R}^n$ a *strong k-parametrization of X* to be a C^k-map $f : (0,1)^m \to \mathbb{R}^n$, $m < n$, with image X, such that $|f^{(\alpha)}(a)| \leqslant 1$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and all $a \in (0,1)^m$. We also define a *hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$* to be the zeroset in $\mathbb{R}^n$ of a nonzero polynomial in $x = (x_1, \dots, x_n)$ over $\mathbb{R}$ of (total) degree $\leqslant e$. The first of these intermediate results is essentially due to Pila and Bombieri, cf. [17, 50].

Theorem 5.2. *Let $n \geqslant 1$ be given. Then for any $e \geqslant 1$ there are $k = k(n,\ e) \geqslant 1$, $\varepsilon = \varepsilon(n,\ e)$, and $c = c(n,\ e)$, such that if $X \subseteq \mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^{ε} many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q}, T)$, with $\varepsilon(n,\ e) \to 0$ as $e \to \infty$.*

We prove this in Section 6.1. In Section 5.2 we obtain Theorem 5.1 from Theorem 5.2 by induction on n, using a strong parametrization result. Yomdin [64] and Gromov [39] proved such a strong parametrization uniformly for the members of any semialgebraic family of subsets of $[-1,1]^n$. We need this for any definable "o-minimal" family. To make this precise, let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$. For $s \in E$,

$$X(s) \ := \ \{a \in \mathbb{R}^n : \ (s,a) \in X\} \quad \text{(a } \textit{section} \text{ of } X\text{)}$$

We consider E, X as describing the family $\big(X(s)\big)_{s \in E}$ of sections $X(s) \subseteq \mathbb{R}^n$; the sets $X(s)$ are the members of the family. If E and X are definable in the o-minimal expansion $\widetilde{\mathbb{R}}$ of $\mathbb{R}$, then its members are definable in $\widetilde{\mathbb{R}}$.

Theorem 5.3. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of $\mathbb{R}$ and $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ with $n \geqslant 1$ definable in $\widetilde{\mathbb{R}}$ such that every section $X(s)$ is a subset of $[-1,1]^n$ with empty interior. Then there is for every $k \geqslant 1$ an $M \in \mathbb{N}$ such that every section $X(s)$ is the union of at most M subsets, each having a strong k-parametrization.*

This is enough for use in the next section, but Chapter 7 gives more precise results. The M in Theorem 5.3 is a source of ineffectivity, and in this connection we call attention to [13, 14, 16] where among other things Binyamini, Novikov and others establish better (logarithmic) bounds for certain o-minimal expansions of $\mathbb{R}$.

5.2 Proof from the two ingredients

Throughout this section we assume $n \geqslant 1$. We begin by stating some elementary facts about X^{alg} and X^{tr} for $X \subseteq \mathbb{R}^n$. The first is obvious:

Lemma 5.4. *If $X = X_1 \cup \cdots \cup X_m$, then $X^{\mathrm{alg}} \supseteq X_1^{\mathrm{alg}} \cup \cdots \cup X_m^{\mathrm{alg}}$, and thus*

$$X^{\mathrm{tr}} \ \subseteq \ X_1^{\mathrm{tr}} \cup \cdots \cup X_m^{\mathrm{tr}}.$$

Note also that if X is open in $\mathbb{R}^n$, then $X^{\mathrm{tr}} = \emptyset$.

Lemma 5.5. *Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and f maps the set $X \subseteq S$ homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Then $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$ and thus $f(X^{\mathrm{tr}}) = Y^{\mathrm{tr}}$. (We allow $m = 0$ for later inductions.)*

Proof. It is clear that $f(X^{\text{alg}}) \subseteq Y^{\text{alg}}$. Also, for any connected infinite semialgebraic set $C \subseteq Y$, the set $f^{-1}(C) \subseteq S$ is semialgebraic (since C and f are), contained in X (since f is injective), hence connected and infinite, and thus $f^{-1}(C) \subseteq X^{\text{alg}}$. This shows $f^{-1}(Y^{\text{alg}}) \subseteq X^{\text{alg}}$, and thus $f(X^{\text{alg}}) = Y^{\text{alg}}$. □

In order to apply Theorem 5.3 we need to reduce to the case of subsets of $[-1,1]^n$. This is done as follows. For $X \subseteq \mathbb{R}^n$ and $I \subseteq \{1,\dots,n\}$, set

$$X_I \ := \ \{a \in X : \ |a_i| > 1 \text{ for all } i \in I,\ |a_i| \leqslant 1 \text{ for all } i \notin I\}$$

and define the semialgebraic map $f_I : \mathbb{R}^n_I \to \mathbb{R}^n$ by $f_I(a) = b$ where $b_i := a_i^{-1}$ for $i \in I$ and $b_i := a_i$ for $i \notin I$. Thus f_I maps $\mathbb{R}^n_I$ homeomorphically onto its image, a subset of $[-1,1]^n$. If $I = \emptyset$, then f_I is the inclusion map $\mathbb{R}^n_I = [-1,1]^n \to \mathbb{R}^n$. Note that for $a \in \mathbb{Q}^n$ we have $f_I(a) \in \mathbb{Q}^n$ and $\mathrm{H}(a) = \mathrm{H}\big(f_I(a)\big)$. Moreover, X is the disjoint union of the sets X_I, and for $Y_I = f_I(X_I)$ we have $Y_I \subseteq [-1,1]^n$, $Y_I^{\text{tr}} = f_I(X_I^{\text{tr}})$ by Lemma 5.5, so $\mathrm{N}(Y_I^{\text{tr}}, T) = \mathrm{N}(X_I^{\text{tr}}, T)$ for all T.

The sketch below actually proves the Counting Theorem, modulo a uniformity assumption that arises at the end of the sketch. This motivates a stronger "definable family" version of the theorem, which we then prove as in the sketch.

In the rest of this section we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\mathbb{R}$, and *definable* is with respect to $\widetilde{\mathbb{R}}$. We exploit facts about semialgebraic cells $C \subseteq \mathbb{R}^n$ and the corresponding homeomorphisms $p_C : C \to p(C)$; see the subsections "Cells" and "Cell Decomposition" of section 4.2.

Sketch of the proof of Theorem 5.1 from Theorems 5.2 and 5.3. Let $X \subseteq \mathbb{R}^n$ be definable. We need to show that there are 'few' rational points on X outside X^{alg}. We proceed by induction on n. By Lemma 5.4 and the remark following it we can remove the interior of X in $\mathbb{R}^n$ from X and arrange that X has empty interior. As indicated just before this sketch we also arrange $X \subseteq [-1,1]^n$.

Let ε be given, and take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 5.2, and take $k = k(n,e)$. Theorem 5.3 gives $M \in \mathbb{N}$ such that X is a union of at most M subsets, each admitting a strong k-parametrization. Then Theorem 5.2 gives $X(\mathbb{Q},T) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=1}^{J} H_{ij}$, where the H_{ij} are hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$, and $J \in \mathbb{N}$, $J \leqslant cT^{\varepsilon/2}$, $c = c(n,e)$ as in that theorem. If $a \in X^{\text{tr}}(\mathbb{Q},T)$ and $a \in H_{ij}$, then clearly $a \in (X \cap H_{ij})^{\text{tr}}$. Thus

$$X^{\text{tr}}(\mathbb{Q},T) \ \subseteq \ \bigcup_{i=1}^{M} \bigcup_{j=1}^{J} (X \cap H_{ij})^{\text{tr}}(\mathbb{Q},T).$$

Let H be any hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$. We aim for an upper bound on $\mathrm{N}\big((X \cap H)^{\text{tr}}, T\big)$ of the form $c_1 T^{\varepsilon/2}$ with $c_1 \in \mathbb{R}^{>}$ independent of H and T. (If we achieve this, then applying this to the hypersurfaces H_{ij} we obtain

$$\mathrm{N}(X^{\text{tr}},T) \ \leqslant \ MJc_1T^{\varepsilon/2} \ \leqslant \ M \cdot cT^{\varepsilon/2} \cdot c_1 T^{\varepsilon/2} \ = \ Mcc_1 \cdot T^{\varepsilon},$$

and we are done.) Take semialgebraic cells $C_1,\dots,C_L$ in $\mathbb{R}^n$, $L \in \mathbb{N}$, such that

$$H \ = \ C_1 \cup \cdots \cup C_L.$$

Suppose $C = C_l$ is one of those cells. Then we have a semialgebraic homeomorphism $p = p_C : C \to p(C) = p(C_l)$ onto an open cell $p(C_l)$ in $\mathbb{R}^{n_l}$ with $n_l < n$, and so p maps $X \cap C_l$ homeomorphically onto its image $Y_l \subseteq p(C_l) \subseteq \mathbb{R}^{n_l}$. Now p is given by omitting $n - n_l$ of the coordinates, so for $a \in C_l(\mathbb{Q})$ we have $p(a) \in \mathbb{Q}^{n_l}$

and $\mathrm{H}\big(p(a)\big) \leqslant \mathrm{H}(a)$. The hypersurfaces of degree $\leqslant e$ in $\mathbb{R}^n$ belong to a single semialgebraic family, so by Proposition 4.4 we can (and do) take here $L \leqslant L(e,n)$, with $L(e,n) \in \mathbb{N}^{\geqslant 1}$ depending only on e, n. By Lemma 5.4,

$$(X \cap H)^{\mathrm{tr}} \ \subseteq \ (X \cap C_1)^{\mathrm{tr}} \cup \cdots \cup (X \cap C_L)^{\mathrm{tr}}.$$

Since the $n_l < n$ we can (and do) assume inductively that for all T,

$$\mathrm{N}(Y_l^{\mathrm{tr}}, T) \ \leqslant \ B_l T^{\varepsilon/2}, \quad l = 1, \ldots, L$$

with $B_l \in \mathbb{R}^{>}$ independent of T. Hence for all T,

$$\mathrm{N}\big((X \cap C_l)^{\mathrm{tr}}), T\big) \ \leqslant \ B_l T^{\varepsilon/2}, \quad l = 1, \ldots, L$$

by Lemma 5.5 applied to the maps $p = p_{C_l}$, and thus

$$\mathrm{N}\big((X \cap H)^{\mathrm{tr}}, T\big) \ \leqslant \ (B_1 + \cdots + B_L) T^{\varepsilon/2}.$$

Assume we can take $B_1, \ldots, B_L \leqslant B$ with $B \in \mathbb{R}^{>}$ depending only on X, ε, not on $H, Y_1, \ldots, Y_L$. Then $c_1 := L(e,n)B$ is a positive real number as aimed for.

The above sketch is a proof, modulo the assumption at the end. The hypersurfaces H in the sketch belong fortunately to a single semialgebraic family, a fact we already used, and so the sets Y_l as H varies can be taken to belong to a single definable family. The inductive hypothesis should accordingly include this uniformity, and so the full proof should be carried out not just for one set X, but uniformly for all sets from a definable family, with constants depending only on the family. This is why we need Theorem 5.3 not just for a single definable $X \subseteq [-1,1]^n$ but for all members of a definable family of such sets. (As to the M introduced at the beginning of the sketch, Theorem 5.3 also provides an M that works for all members of the family.) Below we carry out the details.

The next lemma is a routine consequence of Theorem 4.2 and Proposition 4.4. The $\boldsymbol{i}$-cells in this lemma and the projection maps $p_{\boldsymbol{i}} : \mathbb{R}^n \to \mathbb{R}^d$ in the proof of Theorem 5.7 are defined in the subsection "Cells" from Section 4.2.

Lemma 5.6. *Let $e \geqslant 1$ and set $D := \binom{e+n}{n}$, the dimension of the $\mathbb{R}$-linear space of polynomials over $\mathbb{R}$ in n variables and of degree $\leqslant e$. Then there are $L \in \mathbb{N}^{\geqslant 1}$ and semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \ldots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$, $F := \mathbb{R}^D \setminus \{0\}$, such that*

$$\{\mathcal{H}(t) : \ t \in F\} = \ \textit{set of hypersurfaces in } \mathbb{R}^n \textit{ of degree} \leqslant e,$$

$\mathcal{H}(t) = \mathcal{C}_1(t) \cup \cdots \cup \mathcal{C}_L(t)$ for all $t \in F$, and for each $l \in \{1, \ldots, L\}$ there is an $\boldsymbol{i} = (i_1, \ldots, i_n)$ in $\{0,1\}^n$, $\boldsymbol{i} \neq (1, \ldots, 1)$, with the property that every $\mathcal{C}_l(t)$ with $t \in F$ is a semialgebraic $\boldsymbol{i}$-cell in $\mathbb{R}^n$ or empty.

Two family versions of the Counting theorem. In this subsection we assume that $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

Theorem 5.7. *Let any ε be given. Then there is a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T we have $\mathrm{N}(X(s)^{\mathrm{tr}}, T) \leqslant cT^{\varepsilon}$.*

Proof. We proceed by induction on n. As in the sketch we reduce to the case where $X(s)$ is for every $s \in E$ a subset of $[-1,1]^n$ with empty interior. Take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 5.2, and set $k = k(n,e)$. So for every $Z \subseteq \mathbb{R}^n$ with a strong k-parameterization we can cover $Z(\mathbb{Q},T)$ with at most $cT^{\varepsilon/2}$ hypersurfaces of degree $\leqslant e$ where $c = c(n,e)$ is as in Theorem 5.2. Theorem 5.3 gives $M \in \mathbb{N}$ such that each section $X(s)$ is a union of at most M subsets, each admitting a strong k-parametrization. Let $s \in E$, and let H be a hypersurface of degree $\leqslant e$. As in the sketch we see that by our choice of k, e it is enough to show:

$$\mathrm{N}\left((X(s)\cap H)^{\mathrm{tr}}, T\right) \;\leqslant\; c_1 T^{\varepsilon/2}, \text{ for all } T,$$

where $c_1 \in \mathbb{R}^{>}$ depends only on X, ε, not on s, H, T. Below we provide such c_1.

With the present values of e and n, set $D := \binom{e+n}{n}$, $F := \mathbb{R}^D \setminus \{0\}$, and let $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ be as in Lemma 5.6. For $l = 1, \dots, L$, take $\boldsymbol{i}^l = (i^l_1, \dots, i^l_n)$ in $\{0,1\}^n$, not equal to $(1, \dots, 1)$, such that for all $t \in F$ the subset $\mathcal{C}_l(t)$ of $\mathbb{R}^n$ is a semialgebraic $\boldsymbol{i}^l$-cell or empty, so

$$p_{\boldsymbol{i}^l} : \mathbb{R}^n \to \mathbb{R}^{n_l}, \quad n_l := i^l_1 + \cdots + i^l_n < n,$$

maps $\mathcal{C}_l(t)$ homeomorphically onto its image. Then we have for $l = 1, \dots, L$ a definable set $Y_l \subseteq (E \times F) \times \mathbb{R}^{n_l}$ such that for all $(s,t) \in E \times F$,

$$Y_l(s,t) \;=\; p_{\boldsymbol{i}^l}\big(X(s) \cap \mathcal{C}_l(t)\big).$$

Since all $n_l < n$ we can assume inductively that for all $(s,t) \in E \times F$ and all T,

$$\mathrm{N}\left(Y_l(s,t)^{\mathrm{tr}}, T\right) \;\leqslant\; B_l T^{\varepsilon/2}, \quad l = 1, \dots, L$$

with $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^{>}$ independent of s, t, T. Since $H = \mathcal{H}(t)$ for some $t \in F$,

$$\mathrm{N}\left((X(s)\cap H)^{\mathrm{tr}}, T\right) \;\leqslant\; (B_1 + \cdots + B_L) T^{\varepsilon/2},$$

as in the sketch. Thus $c_1 := B_1 + \cdots + B_L$ is as promised. ☐

Next a variant of Theorem 5.7 where we remove from the sets $X(s)$ only a definable part $V(s)$ of $X(s)^{\mathrm{alg}}$ instead of all of it. The example preceding the statement of Theorem 5.1 shows that this variant is strictly stronger than Theorem 5.7.

Theorem 5.8. *Let any ε be given. Then there is a definable set $V = V(X, \varepsilon) \subseteq X$ and a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T,*

$$V(s) \subseteq X(s)^{\mathrm{alg}} \quad \textit{and} \quad \mathrm{N}\left(X(s) \setminus V(s), T\right) \leqslant cT^{\varepsilon}.$$

Proof. By induction on n. We follow closely the proof of Theorem 5.7. Let $V_0 \subseteq X$ be given by $V_0(s) =$ interior of $X(s)$ in $\mathbb{R}^n$ for $s \in E$. This definable set V_0 will be part of a V as required. Replacing X by $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. We arrange in addition that $X(s) \subseteq [-1,1]^n$ for all $s \in E$. Now take e and $k = k(n,e)$ as in the proof of Theorem 5.7. It will be enough to find a definable $V \subseteq X$ and a constant $c_1 \in \mathbb{R}^{>}$ such that for all $s \in E$, every hypersurface H of degree $\leqslant e$ in $\mathbb{R}^n$, and all T we have

$$V(s) \;\subseteq\; X(s)^{\mathrm{alg}}, \qquad \mathrm{N}\left((X(s)\cap H) \setminus V(s), T\right) \;\leqslant\; c_1 T^{\varepsilon/2}.$$

We take the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \ldots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$ for $l = 1, \ldots, L$ as in the proof of Theorem 5.7. For such l we have $n_l < n$, so we can assume inductively that we have a definable set $W_l \subseteq Y_l$ and a number $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^>$ such that for all $s \in E$, $t \in F$, and T we have

$$W_l(s,t) \;\subseteq\; Y_l(s,t)^{\mathrm{alg}} \quad \text{and} \quad N\big(Y_l(s,t) \setminus W_l(s,t), T\big) \;\leqslant\; B_l T^{\varepsilon/2}.$$

It is now easy to check that the definable set $V \subseteq X$ such that for all $s \in E$,

$$V(s) \;=\; \bigcup_{l=1}^{L} \bigcup_{t \in F} \mathcal{C}_l(t) \cap p_{\boldsymbol{i}^l}^{-1}\big(W_l(s,t)\big)$$

has the desired property. $\square$

In the next two chapters we establish the results used in the proofs above, namely Theorems 5.2 and 5.3. In Chapter 8 we strengthen and extend Theorem 5.8 in several ways without changing the basic inductive set-up of its proof.

CHAPTER 6

The Bombieri-Pila determinant method

This chapter is devoted to establishing Theorem 6.6, which is more precise form of Theorem 5.2. The material here essentially follows the technology around [50, Proposition 4.2].

6.1 Proof of Theorem 5.2

We begin with introducing a key determinant. Let $\boldsymbol{k}$ be a field and set

$$D(n,\ e)\ :=\ \binom{e+n}{n}\ =\ \#\{\alpha \in \mathbb{N}^n :\ |\alpha|\ \leqslant\ e\} \in \mathbb{N}^{\geqslant 1},$$

the dimension of the $\boldsymbol{k}$-linear space of n-variable polynomials over $\boldsymbol{k}$ of (total) degree at most e. Thus $D(n,\ 0) = 1$, $D(n,e) = \frac{e^n}{n!}\big(1+o(1)\big)$ as $e \to \infty$, and if $n \geqslant 1$, then $D(n,e)$ is strictly increasing as a function of e.

For now we fix n and e, set $D := D(n,e)$ and let α range over $\mathbb{N}^n$. By a *hypersurface in* $\boldsymbol{k}^n$ *of degree* $\leqslant e$ we mean the set of zeros in $\boldsymbol{k}^n$ of a nonzero n-variable polynomial of degree $\leqslant e$ with coefficients in $\boldsymbol{k}$.

Lemma 6.1. *A set $S \subseteq \boldsymbol{k}^n$ is contained in some hypersurface in $\boldsymbol{k}^n$ of degree at most e if and only if* $\det(a_i^\alpha)_{|\alpha|\leqslant e, i=1,\ldots,D} = 0$ *for all* $a_1, \ldots, a_D \in S$.

Proof. Let $f = \sum_{|\alpha|\leqslant e} c_\alpha x^\alpha$ be a nonzero polynomial in $x = (x_1, \ldots, x_n)$ of degree at most e with coefficients $c_\alpha \in \boldsymbol{k}$ such that $f = 0$ on S. Then for any points $a_1, \ldots, a_D \in S$ we have $f(a_1) = \cdots = f(a_D) = 0$, that is,

$$\sum_{|\alpha|\leqslant e} c_\alpha\big(a_1^\alpha, \ldots, a_D^\alpha\big)\ =\ 0\ \text{ in }\ \boldsymbol{k}^D,$$

so the D vectors $(a_1^\alpha, \ldots, a_D^\alpha)$ ($|\alpha| \leqslant e$) in the $\boldsymbol{k}$-linear space $\boldsymbol{k}^D$ are linearly dependent, which gives the desired conclusion about the determinant.

Conversely, suppose $\det(a_i^\alpha)_{|\alpha|\leqslant e, i=1,\ldots,D} = 0$ for all $a_1, \ldots, a_D \in S$. Then for $A := \{\alpha : |\alpha| \leqslant e\}$, the $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}^A$ spanned by the vectors $(a^\alpha)_{|\alpha|\leqslant e}$ with $a \in S$ has dimension $< D$. Take $a_1, \ldots, a_M \in S$ such that

$$(a_1^\alpha)_{|\alpha|\leqslant e},\ \ldots,\ (a_M^\alpha)_{|\alpha|\leqslant e}$$

is a basis of this subspace. Then $M < D$, so we have $c_\alpha \in \boldsymbol{k}$ for $|\alpha| \leqslant e$, with $c_\alpha \neq 0$ for some α and $\sum_{|\alpha|\leqslant e} c_\alpha a^\alpha = 0$ for $a = a_1, \ldots, a_M$, and thus for all $a \in S$. ☐

Next we introduce some numbers related to $D = D(n,e)$:

$$E(n,\ e)\ :=\ \binom{e+n-1}{n-1}\ =\ \#\{\alpha:\ |\alpha| = e\},$$

the dimension of the $\boldsymbol{k}$-linear space of homogeneous n-variable polynomials of degree e over $\boldsymbol{k}$. (Here $\binom{-1}{-1} := 1$ and $\binom{k}{-1} := 0$.) So $D(n,e) = \sum_{i=0}^{e} E(n,i)$. Next, we set $V(n,e) := \sum_{i=0}^{e} iE(n,i)$. Now for $i \geqslant 1$,

$$\begin{aligned}
iE(n,i) &= i\binom{i+n-1}{n-1} = n\binom{i+n-1}{n} = nE(n+1,i-1), \quad \text{so} \\
V(n,e) &= n\sum_{i=1}^{e} E(n+1,i-1) = nD(n+1,e-1) \text{ for } e \geqslant 1, \quad V(n,0) = 0,
\end{aligned}$$

and thus for fixed n we have $V(n,e) = \frac{ne^{n+1}}{(n+1)!}\big(1+o(1)\big)$ as $e \to \infty$.

Let $e,m,n \geqslant 1$ below and define $b = b(m,\ n,\ e) \in \mathbb{N}$ by requiring

$$D(m,\ b)\ \leqslant\ D(n,\ e)\ <\ D(m,\ b+1)\ .$$

Next, we set for $b = b(m,n,e)$:

$$\begin{aligned}
B(m,\ n,\ e)\ &:=\ \sum_{i=0}^{b} iE(m,i) + (b+1)\cdot\Big(D(n,\ e) - \sum_{i=0}^{b} E(m,i)\Big) \\
&=\ V(m,b) + (b+1)\cdot\big(D(n,e) - D(m,b)\big) \in \mathbb{N}^{\geqslant 1}, \\
\varepsilon(m,\ n,\ e)\ &:=\ \frac{mneD(n,e)}{B(m,n,e)}.
\end{aligned}$$

Lemma 6.2. *With fixed $m,n \geqslant 1$ and $e \to \infty$, we have:*

1. $b(m,\ n,\ e)\ =\ \left(\frac{m!e^n}{n!}\right)^{1/m}\big(1+o(1)\big)$*;*
2. $B(m,\ n,\ e)\ =\ \frac{m}{(m+1)!}\left(\frac{m!}{n!}\right)^{(m+1)/m} e^{n(m+1)/m}\big(1+o(1)\big)$*;*
3. *if $m < n$, then $\varepsilon(m,\ n,\ e) \to 0$.*

Proof. As to (1), for $e \to \infty$ we have $b = b(m,n,e) \to \infty$, so

$$D(m,b)\ =\ \frac{b^m}{m!}\big(1+o(1)\big)\ \leqslant\ \frac{e^n}{n!}\big(1+o(1)\big)\ \leqslant\ \frac{(b+1)^m}{m!}\big(1+o(1)\big),$$

but the last term here is also $\frac{b^m}{m!}\big(1+o(1)\big)$, like the first term, and this easily yields the asymptotics claimed for b. For (2), substituting the result of (1) in the asymptotics for $D(m,b)$ as $b \to \infty$ leads to $(b+1)\cdot\big(D(n,e) - D(m,b)\big) = o\big(e^{n(m+1)/m}\big)$, and then in the asymptotics for $V(m,b)$ yields the asymptotics claimed for $B(m,n,e)$. Now (3) is an easy consequence of (2). $\square$

In the proof of Proposition 6.4 below we need a reasonable bound on the absolute value of the determinant of a certain $(D \times D)$-matrix of the form $\big(a_i^{\alpha}\big)_{|\alpha| \leqslant e, i=1,\ldots,D}$. We achieve this by expressing the matrix as a sum of simpler matrices. In this connection we need a useful expression for the determinant of a sum of matrices.

Turning to this, let $N \in \mathbb{N}$ and consider an $(N \times N)$-matrix $a = (a_{\mu\nu})_{1 \leqslant \mu,\nu \leqslant N}$ over a field $\boldsymbol{k}$. The determinant of an $(N \times N)$-matrix over $\boldsymbol{k}$ is an alternating multilinear function of its columns. The columns of a are $a_1, \dots, a_N \in \boldsymbol{k}^N$ where $a_\nu = (a_{1\nu}, \dots, a_{N\nu})^{\mathrm{t}} \in \boldsymbol{k}^N$ is the νth column of a. Thus

$$a \ = \ (a_1, \dots, a_N) \in \boldsymbol{k}^N \times \cdots \times \boldsymbol{k}^N \ \text{(with } N \text{ factors } \boldsymbol{k}^N\text{)}.$$

Next, let $a = a^1 + \cdots + a^r$ with $r \in \mathbb{N}$ and $a^1, \dots, a^r$ also $(N \times N)$-matrices over $\boldsymbol{k}$, with a^j having ν^{th} column a^j_ν. Then

$$\begin{aligned} \det a \ &= \ \det\big(a_1, \dots, a_N\big) \ = \ \det\Big(\sum_{j=1}^r a^j_1, \dots, \sum_{j=1}^r a^j_N\Big) \\ &= \ \sum_{\boldsymbol{j}} \det\big(a^{j_1}_1, \dots, a^{j_N}_N\big) \end{aligned}$$

where $\boldsymbol{j} = (j_1, \dots, j_N)$ ranges here and below over elements of $\{1, \dots, r\}^N$. Let $\boldsymbol{j}$ be given. If for some j in $\{1, \dots, r\}$ the number of $\nu \in \{1, \dots, N\}$ with $j_\nu = j$ is more than $\operatorname{rank} a^j$, then the column vectors $a^{j_1}_1, \dots, a^{j_N}_N$ are $\boldsymbol{k}$-linearly dependent, so $\det\big(a^{j_1}_1, \dots, a^{j_N}_N\big) = 0$. Thus if $J \subseteq \{1, \dots, r\}^N$ contains all $\boldsymbol{j}$ such that

$$\#\{\nu \in \{1, \dots, N\} : \ j_\nu = j\} \ \leqslant \ \operatorname{rank} a^j, \ \text{for } j = 1, \dots, r,$$

then

$$(*) \qquad \det a \ = \ \sum_{\boldsymbol{j} \in J} \det\big(a^{j_1}_1, \dots, a^{j_N}_N\big) \ = \ \sum_{\boldsymbol{j} \in J} \det\big(a^{j_\nu}_{\mu\nu}\big)_{1 \leqslant \mu, \nu \leqslant N} \qquad .$$

We shall also use the following observation:

Lemma 6.3. *Let A be a set and V a finite-dimensional subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^A$. Then for any $N \in \mathbb{N}$, functions $f_1, \dots, f_N \in V$, and points $a_1, \dots, a_N$ in A, the rank of the $(N \times N)$-matrix $\big(f_\mu(a_\nu)\big)_{1 \leqslant \mu, \nu \leqslant N}$ over $\boldsymbol{k}$ is $\leqslant \dim V$.*

Proof. The map $f \mapsto \big(f(a_1), \dots, f(a_N)\big) : V \to \boldsymbol{k}^N$ is $\boldsymbol{k}$-linear, so the image of this map is a subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^N$ of dimension $\leqslant \dim V$. ☐

Recall our norm $|(t_1, \dots, t_m)| := \max\{|t_1|, \dots, |t_m|\}$ on $\mathbb{R}^m$ for $m \geqslant 1$.

Proposition 6.4. *Let $e, m, n \geqslant 1$, $m < n$, and $k := b(m, n, e) + 1$. Then there is a constant $K = K(m, n, e)$ with the following property: if $f : (0,1)^m \to \mathbb{R}^n$ is a strong k-parametrization, $0 < r \leqslant 1$, and $a_0, \dots, a_D \in (0,1)^m$ with $D = D(n, e)$ are such that $|a_i - a_0| \leqslant r$ for $i = 1, \dots, D$, then*

$$|\det\big(f(a_i)^\alpha\big)_{|\alpha| \leqslant e, \ i = 1, \dots, D}| \ < \ K r^{B(m,n,e)}.$$

Proof. Let $f = (f_1, \dots, f_n)$ with $f_j : (0,1)^m \to \mathbb{R}$. Taylor expansion of f_j of order $b := b(m, n, e)$ around a_0 with explicit remainder gives for $a \in (0,1)^m$ (and α, β ranging over $\mathbb{N}^m$, $k = b + 1$):

$$\begin{aligned} f_j(a) \ &= \ \sum_{|\alpha| \leqslant b} \frac{f^{(\alpha)}_j}{\alpha!} (a - a_0)^\alpha + \sum_{|\beta| = k} R_{\beta, j}(a)(a - a_0)^\beta, \\ \text{where } \ R_{\beta, j} \ &= \ \frac{|\beta|}{\beta!} \int_0^1 (1 - t)^b f^{(\beta)}_j\big(a_0 + t(a - a_0)\big) dt \ \text{ for } \ |\beta| = k. \end{aligned}$$

Thus for $i = 1, \dots, D$ and $j = 1, \dots, n$:

$$f_j(a_i) \;=\; P_j(a_i - a_0) + R_{ij}(a_i - a_0)$$

where $P_j \in \mathbb{R}[x_1, \dots, x_m]$ has degree $\leqslant b$, the remainder is given by a homogeneous polynomial $R_{ij} \in \mathbb{R}[x_1, \dots, x_m]$ of degree $k = b + 1$, and all coefficients of P_j and R_{ij} are bounded in absolute value by 1. Let $|\alpha| \leqslant e$. Then for $i = 1, \dots, D$ we have

$$\prod_{j=1}^{n} (P_j + R_{ij})^{\alpha_j} \;=\; P_\alpha + R_{i\alpha}$$

with $P_\alpha \in \mathbb{R}[x_1, \dots, x_m]$ of degree $\leqslant b$, the remainder $R_{i\alpha} \in \mathbb{R}[x_1, \dots, x_m]$ has only monomials of degree $> b$, and every coefficient of P_α and $R_{i\alpha}$ is bounded in absolute value by $D(m,k)^{|\alpha|}$, the latter because $\prod_{j=1}^{n}(P_j + R_{ij})^{\alpha_j}$ is a product of $|\alpha|$ factors of the form $\sum c_\beta x^\beta$, with the summation over the $\beta \in \mathbb{N}^m$ with $|\beta| \leqslant k$, and real coefficients c_β with $|c_\beta| \leqslant 1$. Hence for $i = 1, \dots, D$,

$$f(a_i)^\alpha \;=\; \prod_{j=1}^{n} f_j(a_i)^{\alpha_j} \;=\; P_\alpha(a_i - a_0) + R_{i\alpha}(a_i - a_0).$$

We have $D(m,k)^{|\alpha|} \leqslant D(m,k)^e \leqslant c$ for a positive constant $c = c(m,n,e)$ depending only on m, n, e. Next, $P_\alpha = \sum_{j=0}^{b} P_\alpha^j$ where $P_\alpha^j \in \mathbb{R}[x_1, \dots, x_m]$ is homogeneous of degree j. In the matrix algebra $\mathbb{R}^{D \times D}$ this yields the sum decomposition

$$\begin{aligned}
\big(f(a_i)^\alpha\big)_{\alpha,i} \;&=\; \sum_{j=0}^{b} \big(P_\alpha^j(a_i - a_0)\big)_{\alpha,i} + \big(R_{i\alpha}(a_i - a_0)\big)_{\alpha,i} \\
&=\; \sum_{j=0}^{k} \big(P_{i\alpha}^j(a_i - a_0)\big)_{\alpha,i}
\end{aligned}$$

where $P_{i\alpha}^j := P_\alpha^j$ for $j = 0, \dots, b$ and $P_{i\alpha}^k := R_{i\alpha}$. For $j = 0, \dots, b$ the rank of the matrix $\big(P_{i\alpha}^j(a_i - a_0)\big)_{\alpha,i} = \big(P_\alpha^j(a_i - a_0)\big)_{\alpha,i}$ is at most $E(m,j)$ by Lemma 6.3, so expression $(*)$ for the determinant of such a sum gives

$$\det\big(f(a_i)^\alpha\big)_{\alpha,i} \;=\; \sum_{\boldsymbol{j} \in J} \det\big(P_{i\alpha}^{j_i}(a_i - a_0)\big)_{\alpha,i}$$

where J is the set of all $\boldsymbol{j} = (j_1, \dots, j_D) \in \{0, \dots, b+1\}^D$ such that

$$\#\{\nu \in \{1, \dots, D\} : \; j_\nu = j\} \;\leqslant\; E(m,j), \qquad \text{for } j = 0, \dots, b.$$

Then for $\boldsymbol{j} \in J$ we have $|\det\big(P_{i\alpha}^{j_i}(a_i - a_0)\big)_{\alpha,i}| \leqslant D! c^D r^{|\boldsymbol{j}|}$. It remains to show that for $\boldsymbol{j} \in J$ we have $|\boldsymbol{j}| \geqslant B(m,n,e)$, because then

$$|\det\big(f(a_i)^\alpha\big)_{|\alpha| \leqslant e,\; i=1,\dots,D}| \;\leqslant\; \#J \cdot D! c^D r^{B(m,n,e)},$$

which gives a constant $K = K(m,n,e)$ as claimed.

Fix any $\boldsymbol{j} \in J$, and let $d_j \in \mathbb{N}$ for $j = 0, \ldots, b$ be such that

$$\#\{\nu \in \{1, \ldots, D\} : \ j_\nu = j\} \ = \ E(m, j) - d_j,$$

and set $N := \#\{\nu \in \{1, \ldots, D\} : \ j_\nu = b+1\}$. Then

$$D \ = \ D(n, e) \ = \ \sum_{j=0}^{b} (E(m, j) - d_j) + N \ = \ D(m, b) - \sum_{j=0}^{b} d_j + N,$$

so $N \ = \ D(n, e) - D(m, b) + d$ with $d := \sum_{j=0}^{b} d_j$. Hence

$$\begin{aligned} |\boldsymbol{j}| \ &= \ \sum_{\nu=1}^{D} j_\nu \ = \ \sum_{j=0}^{b} j\big(E(m, j) - d_j\big) + (b+1)N \\ &= \ V(m, b) - \sum_{j=0}^{b} j d_j + (b+1)\big(D(n, e) - D(m, b) + d\big) \\ &= \ V(m, b) + (b+1)\big(D(n, e) - D(m, b)\big) + \sum_{j=0}^{b} (b+1-j) d_j \\ &= \ B(m, n, e) + \sum_{j=0}^{b} (b+1-j) d_j \ \geqslant \ B(m, n, e). \end{aligned}$$

□

Next an observation that allows us to exploit (as Liouville did) the simple fact that if $r \in \mathbb{Z}$ and $|r| < 1$, then $r = 0$.

Lemma 6.5. *Let points $b_1, \ldots, b_D \in \mathbb{Q}^n$ with $D = D(n, e)$ be given such that $\mathrm{H}(b_1), \ldots, \mathrm{H}(b_D) \leqslant t$, where $t \geqslant 1$. Then*

$$\det \big(b_i^\alpha\big)_{|\alpha| \leqslant e, i} \ \in \ \frac{\mathbb{Z}}{s} \quad \text{ with } s \in \mathbb{N}^{\geqslant 1}, \ s \leqslant t^{neD}.$$

Proof. For $i = 1, \ldots, D$ we have $b_i = (b_{i1}, \ldots, b_{in})$ with $b_{ij} = c_{ij}/s_{ij}$, $c_{ij}, s_{ij} \in \mathbb{Z}$, $1 \leqslant s_{ij} \leqslant t$, so

$$b_i^\alpha = \prod_{j=1}^{n} c_{ij}^{\alpha_j} / \prod_{j=1}^{n} s_{ij}^{\alpha_j} \ \in \ \frac{\mathbb{Z}}{s_{i\alpha}}, \quad s_{i\alpha} \ := \ \prod_{j=1}^{n} s_{ij}^{\alpha_j}.$$

Let $\{\alpha : \ |\alpha| \leqslant e\} = \{\alpha_1, \ldots, \alpha_D\}$. Then $\det \big(b_i^\alpha\big)_{|\alpha| \leqslant e, i}$ is a sum of terms of the form $\pm \prod_{i=1}^{D} b_i^{\alpha_{\sigma(i)}}$ where σ is a permutation of $\{1, \ldots, D\}$. Now the term $\pm \prod_{i=1}^{D} b_i^{\alpha_{\sigma(i)}}$ corresponding to σ lies in $\frac{\mathbb{Z}}{s_\sigma}$ with

$$s_\sigma \ := \ \prod_{i=1}^{D} s_{i\alpha_{\sigma(i)}} \ = \ \prod_{i=1}^{D} \prod_{j=1}^{n} s_{ij}^{\alpha_{\sigma(i)j}}$$

and clearly $s := \prod_{i=1}^{D} \prod_{j=1}^{n} s_{ij}^{e}$ is a common integer multiple of the integers s_σ with $1 \leqslant s \leqslant t^{neD}$, so s has the desired property. □

The following is Theorem 5.2 with more explicit values of k and ε.

Theorem 6.6. *Let $e, m, n \geqslant 1$, $m < n$; set $k := b(m, n, e) + 1$, $\varepsilon := \varepsilon(m, n, e)$. Let $X \subseteq \mathbb{R}^n$ have a strong k-parametrization $f : (0, 1)^m \to \mathbb{R}^n$. Then for all T at most cT^ε hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q}, T)$, where $c = c(m, n, e)$ depends only on m, n, e.*

Proof. Let $K = K(m, n, e)$ be as in Proposition 6.4, and let T be given. With $D = D(n, e)$, let $a_1, \ldots, a_D \in (0,1)^m$ be such that $f(a_1), \ldots, f(a_D) \in X(\mathbb{Q}, T)$. Then Lemma 6.5 gives $s \in \mathbb{N}^{\geqslant 1}$ with $s \leqslant T^{neD}$ (so $T^{-neD} \leqslant 1/s$) such that

$$\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e, i=1,\ldots D} \in \frac{\mathbb{Z}}{s}.$$

Assume also that $0 < r \leqslant 1$ and $a_0 \in (0,1)^m$ are such that $|a_i - a_0| \leqslant r$ for $i = 1, \ldots, D$. Can we guarantee that $\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e, i=1,\ldots D} = 0$ if r is small enough? Proposition 6.4 gives

$$|\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e,\ i=1,\ldots,D}| < Kr^B, \quad B = B(m, n, e).$$

So the answer to the question is *yes*: it is enough that $Kr^B \leqslant T^{-neD}$, that is, $r \leqslant \big(K^{-1}T^{-neD}\big)^{1/B}$. Next, considering closed balls of radius r with respect to the norm $|\cdot|$, centered at a point in $(0,1)^m$, how many are enough to cover $(0,1)^m$? For $m = 1$, the interval $(0,1)$ is covered by e segments $[a - r, a + r]$ with $0 < a < 1$, for any natural number e with $2re \geqslant 1$, and there is clearly such an e with $e \leqslant r^{-1}$. Hence at most r^{-m} closed balls of radius r centered at points in $(0,1)^m$ are enough to cover $(0,1)^m$. Taking $r = \big(K^{-1}T^{-neD}\big)^{1/B}$ it follows from Lemma 6.1 that at most $r^{-m} = K^{m/B}T^{mneD/B} = K^{m/B}T^{\varepsilon}$ hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $X(\mathbb{Q}, T)$. So the theorem holds with $c = K^{m/B}$. $\square$

CHAPTER 7

An o-minimal Yomdin-Gromov theorem

Throughout this chapter, R is an o-minimal field. We refer to Section 4.2 for the basic facts about o-minimal fields; in particular on the later subsections in that section concerning differentiation and smoothness. As usual we identify $\mathbb{Q}$ with the prime subfield of R. We drop the subscript R in expressions like $(0,1)_R$ $(=\{t \in R : 0 < t < 1\})$ and $[a,b]_R$ $(=\{t \in R : a \leqslant t \leqslant b\})$ for $a < b$ in R.

7.1 Parametrization

Let $X \subseteq R^m$ be definable. Call X *strongly bounded* if $X \subseteq [-N,N]^m$ for some N in $\mathbb{N}$. Call a definable map $f : X \to R^n$ strongly bounded if its graph $\Gamma(f) \subseteq R^{m+n}$ is strongly bounded; equivalently, $X \subseteq R^m$ and $f(X) \subseteq R^n$ are strongly bounded.

A *partial k-parametrization of* X is a definable C^k-map $f : (0,1)^l \to R^m$ such that $l = \dim X$, the image of f is contained in X, and $f^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$. A *k-parametrization of* X is a finite set of partial k-parametrizations of X whose images cover X; note that then X is strongly bounded. As a trivial example, if X is finite and strongly bounded, then X has the k-parametrization $\{\phi_a : a \in X\}$, where $\phi_a : (0,1)^0 \to R^m$ takes the value a.

The basic ideas for the proofs of the next two parametrization theorems stem from Yomdin [64] and Gromov [39] who considered the semialgebraic case. For our purpose we need to work in an arbitrary o-minimal field.

Theorem 7.1. *Any strongly bounded definable set $X \subseteq R^m$ has for every $k \geqslant 1$ a k-parametrization.*

The inductive proof of this theorem also requires a version for definable maps. A *k-reparametrization* of a definable map $f : X \to R^n$ is a k-parametrization Φ of its domain X such that for every $\phi : (0,1)^l \to R^m$ in Φ, $f \circ \phi$ is of class C^k and $(f \circ \phi)^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$; note that then $\{f \circ \phi : \phi \in \Phi\}$ is a k-parametrization of $f(X)$, provided $\dim X = \dim f(X)$.

Theorem 7.2. *Any strongly bounded definable map $f : X \to R^n$, $X \subseteq R^m$ has for every $k \geqslant 1$ a k-reparametrization.*

Sections 7.2, 7.3, 7.4 contain the proof of Theorems 7.1 and 7.2. In Section 7.4 we assume R is $\aleph_0$-saturated, and thus non-archimedean. This can always be arranged by passing to a suitable elementary extension of R and noting that the statements of 7.1 and 7.2 pull back to the original R. (See Section 4.3 for "$\aleph_0$-saturated" and "elementary extension" and the relevant facts about these notions. See in particular the last two subsections of that section for a more detailed explanation of how these facts apply to proving Theorems 7.1 and 7.2.)

We often use the following, obtained by repeated use of the Chain Rule:

Lemma 7.3. *Let $f : U \to R,\ g : V \to R$ be definable of class C^k, $k \geqslant 1$, with U, V (definable) open subsets of R. Then $f \circ g : V \cap g^{-1}(U) \to R$ is of class C^k with*

$$(f \circ g)^{(k)} \ = \ \sum_{i=1}^{k} (f^{(i)} \circ g) \cdot p_{ik}\big(g^{(1)}, \dots, g^{(k-i+1)}\big)$$

where the $p_{ik} \in \mathbb{Z}[x_1, \dots, x_{k-i+1}]$ have constant term 0 and $p_{kk} = x_1^k$.

Lemma 7.4. *With $U \subseteq R^l$, $V \subseteq R^m$, let $f : U \to R^m$, $g : V \to R^n$ be definable of class C^k such that $f(U) \subseteq V$ and $f^{(\alpha)}$ and $g^{(\beta)}$ are strongly bounded for all $\alpha \in \mathbb{N}^l$ and $\beta \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and $|\beta| \leqslant k$. Then the definable map $g \circ f : U \to R^n$ is of class C^k with strongly bounded $(g \circ f)^{(\alpha)}$ for all $\alpha \in \mathbb{N}^l$ with $|\alpha| \leqslant k$.*

7.2 Reparametrizing unary functions

Much in this section is bookkeeping, but we begin with a key analytic fact:

Lemma 7.5. *Let $f : (0,1) \to R$ be a definable C^k-function, $k \geqslant 2$, with strongly bounded $f^{(j)}$ for $0 \leqslant j \leqslant k-1$ and decreasing $|f^{(k)}|$. Define $g : (0,1) \to R$ by $g(t) = f(t^2)$. Then $g^{(j)}$ is strongly bounded for $0 \leqslant j \leqslant k$.*

Proof. Let t range over $(0,1)$. Lemma 7.3 gives

$$g^{(j)}(t) \ = \ \sum_{i=0}^{j} \rho_{ij}(t).f^{(i)}(t^2), \qquad j = 0, \dots, k$$

where each function ρ_{ij} is given by a 1-variable polynomial with integer coefficients, of degree $\leqslant i$, and with $\rho_{jj}(t) = 2^j t^j$. All summands here are strongly bounded except possibly the one with $i = j = k$, which is $2^k t^k f^{(k)}(t^2)$. So it suffices that $t^k f^{(k)}(t^2)$ is strongly bounded. Let $c \in \mathbb{Q}^{>0}$ be a strong bound for $f^{(k-1)}$. We claim that then $|f^{(k)}(t)| \leqslant 4c/t$ for all t. Suppose towards a contradiction that $t_0 \in (0,1)$ is a counterexample, that is, $|f^{(k)}(t_0)| > 4c/t_0$. Then the Mean Value Theorem (Lemma 4.11) provides a $\xi \in [t_0/2, t_0]$ such that

$$f^{(k-1)}(t_0) - f^{(k-1)}(t_0/2) \ = \ f^{(k)}(\xi).(t_0 - t_0/2) \ = \ f^{(k)}(\xi) \cdot t_0/2.$$

Since $|f^{(k)}|$ is decreasing by assumption, $|f^{(k)}(\xi)| \ \geqslant \ |f^{(k)}(t_0)| > 4c/t_0$. Hence

$$2c \ \geqslant \ |f^{(k-1)}(t_0) - f^{(k-1)}(t_0/2)| \ > \ (4c/t_0) \cdot (t_0/2) \ = \ 2c.$$

This contradiction proves our claim. Then for all t,

$$|t^k f^{(k)}(t^2)| \ \leqslant \ t^k \cdot (4c/t^2) \ = \ 4ct^{k-2} \ \leqslant \ 4c$$

using $k \geqslant 2$ for the last inequality. □

The lemma fails for $k = 1$, with $t \mapsto t^{1/3}$ as a counterexample.

Lemma 7.6. *Let $f : (0,1) \to R$ be definable and strongly bounded. Then f has a 1-reparametrization Φ such that for every $\phi \in \Phi$, ϕ or $f \circ \phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. Take elements $a_0 = 0 < a_1 < \cdots < a_n < a_{n+1} = 1$ in R such that, for $i = 0, 1, \ldots, n$, f is of class C^1 on (a_i, a_{i+1}), and either $|f'| \leqslant 1$ on (a_i, a_{i+1}), or $|f'| > 1$ on (a_i, a_{i+1}). Let $i \in \{0, \ldots, n\}$. If $|f'| \leqslant 1$ on (a_i, a_{i+1}), define

$$\phi_i \ : \ (0,1) \to R, \qquad \phi_i(t) \ := \ a_i + (a_{i+1} - a_i)t.$$

If $|f'| > 1$ on (a_i, a_{i+1}), set

$$b_i \ := \ \lim_{t \downarrow a_i} f(t), \qquad b_{i+1} \ := \ \lim_{t \uparrow a_{i+1}} f(t)$$

and as in this case f is continuous and strictly monotone on (a_i, a_{i+1}) we can define $\phi_i : (0,1) \to R$ by $\phi_i(t) = f^{-1}\big(b_i + (b_{i+1} - b_i)t\big)$, where f^{-1} denotes the compositional inverse of the restriction of f to (a_i, a_{i+1}); this compositional inverse has domain (b_i, b_{i+1}) if $b_i < b_{i+1}$, and domain (b_{i+1}, b_i) if $b_i > b_{i+1}$.

In either case, ϕ_i maps $(0,1)$ onto (a_i, a_{i+1}) and both ϕ_i and $f \circ \phi_i$ are of class C^1 with strongly bounded derivative. Moreover, ϕ_i or $f \circ \phi_i$ is given by a univariate polynomial of degree 1 with strongly bounded coefficients in R. Thus

$$\Phi \ := \ \{\phi_0, \ldots, \phi_n, \hat{a}_1, \ldots, \hat{a}_n\}$$

is a 1-reparametrization of f as required, where $\hat{a}_i$ denotes the constant function on $(0,1)$ with value a_i. $\square$

Lemma 7.7. *Let $k \geqslant 1$ and suppose $f : (0,1) \to R$ is definable and strongly bounded. Then f has a k-reparametrization Φ such that for all $\phi \in \Phi$, ϕ or $f \circ \phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. By induction on k. The case $k = 1$ is Lemma 7.6. Suppose $k \geqslant 2$ and Φ is a $(k-1)$-reparametrization of f with the additional property. Let $\phi \in \Phi$. Then $\{\phi, f \circ \phi\} = \{g, h\}$ where g is given by a univariate polynomial with strongly bounded coefficients in R. Thus g is of class C^∞, and $g^{(i)}$ is strongly bounded for all $i \in \mathbb{N}$, and h is of class C^{k-1} with strongly bounded $h^{(j)}$ for $j = 0, \ldots, k-1$. In order to apply Lemma 7.5 we use o-minimality: take elements

$$a_0 \ = \ 0 \ < \ a_1 \ < \ \ldots \ < \ a_{n_\phi} \ < \ a_{n_\phi + 1} \ = \ 1$$

in R such that for $i = 0, \ldots, n_\phi$, the function h is of class C^k on (a_i, a_{i+1}) and $|h^{(k)}|$ is monotone on (a_i, a_{i+1}). Define $\theta_{\phi,i} : (0,1) \to R$ as $t \mapsto a_i + (a_{i+1} - a_i)t$, if $|h^{(k)}|$ is decreasing, and as $t \mapsto a_{i+1} + (a_i - a_{i+1})t$, otherwise; so $\theta_{\phi,i}$ has image (a_i, a_{i+1}). Then $h \circ \theta_{\phi,i} : (0,1) \to R$ is of class C^k, $(h \circ \theta_{\phi,i})^{(j)}$ is strongly bounded for $j = 0, \ldots, k-1$, and $|h \circ \theta_{\phi,i}^{(k)}|$ is decreasing. Let $\rho : (0,1) \to (0,1)$ be the C^∞-bijection sending t to t^2. By Lemma 7.5, the definable C^k-function $h \circ \theta_{\phi,i} \circ \rho : (0,1) \to R$ has strongly bounded jth derivative for $j = 0, \ldots, k$. The function $g \circ \theta_{\phi,i} \circ \rho$ is still given by a 1-variable polynomial with strongly bounded coefficients in R, and $\{g \circ \theta_{\phi,i} \circ \rho, h \circ \theta_{\phi,i} \circ \rho\} = \{\phi \circ \theta_{\phi,i} \circ \rho, f \circ (\phi \circ \theta_{\phi,i} \circ \rho)\}$. The images of the functions $\phi \circ \theta_{\phi,i} \circ \rho$ with $i \in \{0, \ldots, n_\phi\}$ cover the image of ϕ apart from finitely many points. So adding finitely many constant functions with domain $(0,1)$ and values in $(0,1)$ to the set $\{\phi \circ \theta_{\phi,i} \circ \rho : \ \phi \in \Phi, \ i = 0, \ldots, n_\phi\}$ we obtain a k-reparametrization of f as claimed in the statement of the lemma. $\square$

Corollary 7.8. *Let $f : X \to R$ be definable and strongly bounded with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. The case that X is finite is obvious. Suppose X is infinite, $k \geqslant 1$. Since X is a finite union of strongly bounded intervals and points, it has a k-parametrization Φ by univariate polynomial functions of degree $\leqslant 1$.

Now Lemma 7.7 provides for every $\phi : (0,1) \to R$ in Φ a k-reparametrization Ψ_ϕ of $f \circ \phi : (0,1) \to R$; then $\{\phi \circ \psi : \ \phi \in \Phi, \ \psi \in \Psi_\phi\}$ is a k-reparametrization of f. ☐

Next one might reparametrize "curves" $(0,1) \to R^n$ with $n \geqslant 2$, but there is nothing special about the univariate case here, so we do the general case:

Lemma 7.9. *Let $k, m \geqslant 1$, and suppose that every strongly bounded definable function $X \to R$ with $X \subseteq R^l$, $l \leqslant m$, has a k-reparametrization. Then every strongly bounded definable map $X \to R^n$ with $X \subseteq R^l$, $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Proof. By induction on $n \geqslant 1$. Suppose $F : X \to R^n$ and $f : X \to R$ with $X \subseteq R^m$ are definable, strongly bounded, and F has a k-reparametrization. It is enough to show that then the strongly bounded definable map $(F, f) : X \to R^{n+1}$ has a k-reparametrization. The case of finite X being trivial, assume X is infinite. Let Φ be a k-reparametrization of F and let $\phi \in \Phi$, $\phi : (0,1)^l \to R^m$, $l = \dim X \leqslant m$. Applying the hypothesis of the lemma to the map $f \circ \phi : (0,1)^l \to R$ we obtain a k-reparametrization Ψ_ϕ of it. Then using Lemma 7.4, $\{\phi \circ \psi : \ \phi \in \Phi, \ \psi \in \Psi_\phi\}$ is a k-reparametrization of (F, f). ☐

Remark. At one point we need a slight variant of this lemma, with the same proof: *Let $k, m \geqslant 1$, and suppose that every strongly bounded definable function $(0,1)^l \to R$ with $l \leqslant m$ has a k-reparametrization. Then every strongly bounded definable map $(0,1)^l \to R^n$ with $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Corollary 7.10. *Let $n \geqslant 1$ and suppose $f : X \to R^n$ is definable and strongly bounded, with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. Immediate from Corollary 7.8 and the case $m = 1$ of Lemma 7.9. ☐

7.3 Convergence

In this section we continue to work with our o-minimal field R. A set $X \subseteq R$ is said to be *bounded* if $X \subseteq [-r, r]$ for some $r \in R^{>}$. Since each definable subset of R is a finite disjoint union of intervals and singletons, we can assign to each bounded definable set $X \subseteq R$ its *length* $\ell(X) \in R$ so that $\ell(X) = b - a$ if X is an interval (a, b), $a < b$ in R, $\ell(X) = 0$ for $X = \{a\}$, $a \in R$, and $\ell(X) = \ell(X_1) + \ell(X_2)$ if X is the disjoint union of definable subsets X_1, X_2.

Let $a, b \in R$, $a < b$, let $f : (a, b) \to R$ be definable and bounded (the latter meaning that image(f) is bounded), and let $L \in R^{>}$. For $s \in (a, b)$ we declare "$|f'(s)| > L$" to mean: f is differentiable at s, and $|f'(s)| > L$. Suppose that $|f'(s)| > L$ for all $s \in (a, b)$. Then by the Mean Value Theorem (Lemma 4.11),

$$\ell\big(\mathrm{image}(f)\big) > L \cdot (b - a).$$

Let $E \subseteq R^m$ be definable and $(f_s)_{s \in E}$ a *definable family* of functions $f_s : (a, b) \to R$, meaning that the function $(s, t) \mapsto f_s(t) : E \times (a, b) \to R$ is definable. We now use the observation above to obtain the following:

Lemma 7.11. *Suppose $N \in \mathbb{N}$ is such that $|f_s(t)| \leqslant N$ for all $s \in E$ and $t \in (a, b)$. Then there is $M \in \mathbb{N}$ such that for all $L \in R^{>}$ and $s \in E$,*

$$\ell\big(\{t \in (a, b) : \ |f_s'(t)| > L\}\big) \ \leqslant \ M/L.$$

Proof. Let $L \in R^{>}$, $s \in E$, and set $X_{L,s} := \{t \in (a,b) : |f_s'(t)| > L\}$. O-minimality (Theorem 4.2, Proposition 4.4, Theorem 4.14) gives a finite bound m independent of L, s, and disjoint intervals $(a_i, b_i) \subseteq X_{L,s}$, $i = 1, \dots, m_{L,s} \leqslant m$, such that $X_{L,s} \setminus \bigcup_{i=1}^{m_{L,s}} (a_i, b_i)$ is finite. By the observation above, $2N > L(b_i - a_i)$ for $i = 1, \dots, m_{L,s}$, so $2m_{L,s}N > L\ell(X_{L,s})$. Thus $\ell(X_{L,s}) < 2mN/L$. $\square$

Some notation and terminology. For definable open $U \subseteq R^{m+1}$, $V \Subset U$ means that V is a definable open subset of R^{m+1} with $V \subseteq U$ and $\dim(U \setminus V) \leqslant m$.

A *cofinite subset of a set* X is a set $X_0 \subseteq X$ such that $X \setminus X_0$ is finite. Let $\pi_{m+1} : R^{m+1} \to R$ be given by $\pi(t_1, \dots, t_m, t_{m+1}) = t_{m+1}$. Note that if $X \subseteq R^{m+1}$ is definable, then so is $\pi_{m+1}(X) \subseteq R$.

Recall: $|y| = \max\{|y_1|, \dots, |y_n|\}$ for y in R^n. For a definable map $f : X \to R^n$ with (necessarily definable) $X \subseteq R^m$, we set

$$\|f\| \ := \ \sup_{a \in X} |f(a)| \in [0, +\infty].$$

Note that if X is nonempty and closed and bounded in R^m and f is continuous, then this supremum is a maximum, by Corollary 4.8.

Lemma 7.12. *Let $f : U \to R$, $U \Subset (0,1)^{m+1}$, be a strongly bounded definable C^1-function. Suppose $\partial f/\partial x_i$ is strongly bounded for $i = 1, \dots, m$. Then*

$$\{t \in \pi_{m+1}(U) : \ \partial f/\partial x_{m+1}(-, t) \text{ is bounded}\}$$

is a cofinite subset of $\pi_{m+1}(U)$, and thus of $(0,1)$.

Proof. Suppose not. Then the set $\{t \in \pi_{m+1}(U) : \partial f/\partial x_{m+1}(-,t) \text{ is unbounded}\}$ contains an interval $(a,b) \subseteq (0,1)$. Definable Selection (Proposition 4.10) then gives a definable family $(\gamma_L)_{L \in R^{>}}$ of maps

$$\gamma_L \ = \ (\gamma_{L,1}, \dots, \gamma_{L,m}) \ : \ (a,b) \to (0,1)^m$$

such that for all $L \in R^{>}$ and $t \in (a,b)$ we have $\big(\gamma_L(t), t\big) \in U$ and

$$\Big|\frac{\partial f}{\partial x_{m+1}}\big(\gamma_L(t), t\big)\Big| \ > \ L.$$

Take $N \in \mathbb{N}$ such that $\|f\| \leqslant N$ and $\|\frac{\partial f}{\partial x_i}\| \leqslant N$ for $i = 1, \dots, m$. Let $f_L : (a,b) \to R$ be given by $f_L(t) := f\big(\gamma_L(t), t\big)$. Applying Lemma 7.11 to the definable families $\big(3mN\gamma_{L,1}\big)_{L \in R^{>}}, \dots, \big(3mN\gamma_{L,m}\big)_{L \in R^{>}}$, $(3f_L)_{L \in R^{>}}$ gives $M \in \mathbb{N}$ with the property that for all $L \in R^{>}$ there is a definable closed subset X_L of (a,b) with $\ell(X_L) \leqslant M/L$ such that for all $t \in (a,b) \setminus X_L$ the map γ_L is differentiable at t and

$$3mN|\gamma_{L,i}'(t)| \ \leqslant \ L, \qquad i = 1, \dots, m, \qquad \text{and}$$

$$|f_L'(t)| \ = \ \Big|\sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\gamma_L(t), t) \cdot \gamma_{L,i}'(t) + \frac{\partial f}{\partial x_{m+1}}(\gamma_L(t), t)\Big| \ \leqslant \ \frac{L}{3}.$$

Now take L with $M/L < b - a$ and $t \in (a,b)$ satisfying the $m+2$ displayed inequalities. The case $m = 0$ gives an immediate contradiction, and for $m \geqslant 1$,

$$\Big|\sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\gamma_L(t), t) \cdot \gamma_{L,i}'(t)\Big| \ \leqslant \ mN \cdot \frac{L}{3mN} \ = \ \frac{L}{3},$$

contradicting the conjunction of the first and last inequality. $\square$

The *normalization* of a function $\psi : I \to R$ on an interval $I = (a,b) \subseteq (0,1)$ is the function $t \mapsto \psi\big((b-a)t+a\big) : (0,1) \to R$; its image is $\psi(I)$.

Notation about "changing the last variable": For $\phi : (0,1) \to R$ we set

$$I_\phi \;:\; (0,1)^{m+1} \to R^{m+1}, \qquad (t_1,\dots,t_m,t_{m+1}) \mapsto \big(t_1,\dots,t_m,\phi(t_{m+1})\big),$$

and for $f : X \to R^n$, $X \subseteq R^{m+1}$ we set

$$f_\phi \;:=\; f \circ I_\phi \;:\; (I_\phi)^{-1}(X) \to R^n, \quad (t_1,\dots,t_m,t_{m+1}) \mapsto f\big(t_1,\dots,t_m,\phi(t_{m+1})\big).$$

Lemma 7.13. *Let $f : U \to R$, $U \Subset (0,1)^{m+1}$, be a strongly bounded definable C^1-function such that $\partial f/\partial x_i$ is strongly bounded for $i = 1,\dots,m$. Then there is for each $k \geqslant 1$ a k-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi/\partial x_i$ is strongly bounded on V, for $i = 1,\dots,m+1$.*

Proof. Lemma 7.12 yields a finite $F \subseteq \pi_{m+1}(U)$ such that $\frac{\partial f}{\partial x_{m+1}}(-,t)$ is bounded, for every $t \in \pi_{m+1}(U) \setminus F$. Set $V_0 := U \setminus \pi_{m+1}^{-1}(F)$, so $V_0 \Subset U$ and $\pi_{m+1}(V_0) = \pi_{m+1}(U) \setminus F$. For each $t \in \pi_{m+1}(V_0)$ we take a point $a = a(t) \in (0,1)^m$ such that

$$(a,t) \in V_0, \qquad \|\frac{\partial f}{\partial x_{m+1}}(-,t)\| \;\leqslant\; 2|\frac{\partial f}{\partial x_{m+1}}(a,t)|.$$

We arrange by Proposition 4.10 that $t \mapsto a(t) : \pi_{m+1}(V_0) \to R^m$ is definable. Let $\gamma : \pi_{m+1}(V_0) \to V_0$ be defined by $\gamma(t) = \big(a(t),t\big)$. Let $k \geqslant 1$. Corollary 7.10 gives a k-reparametrization Φ_0 of the map

$$g \;:\; \pi_{m+1}(V_0) \to R^{m+2}, \quad t \mapsto \big(\gamma(t), f(\gamma(t))\big).$$

We now change V_0, Φ_0 to V, Φ as follows. The Monotonicity Theorem 4.1 yields for each $\phi \in \Phi_0$ a finite partition $\mathcal{P}_\phi$ of its domain $(0,1)$ into subintervals and singletons such that on each interval in $\mathcal{P}_\phi$ the function ϕ is either constant or strictly monotone. First, replace each $\phi \in \Phi_0$ by the restrictions of ϕ to those intervals in $\mathcal{P}_\phi$ on which ϕ is strictly monotone. Next, replace each of those restrictions with its normalization. The resulting set Φ of (strictly monotone) functions is still a k-parametrization of a cofinite subset of $\pi_{m+1}(V_0)$. Now set

$$V \;:=\; V_0 \cap \bigcap_{\phi\in\Phi} I_\phi^{-1}(U) \;=\; V_0 \setminus \bigcup_{\phi\in\Phi} I_\phi^{-1}[(0,1)^{m+1} \setminus U].$$

The injectivity and continuity of the $\phi \in \Phi$ gives $V \Subset U$. Let $\phi \in \Phi$. Then $I_\phi(V) \subseteq U$, so f_ϕ is of class C^1 on V in view of $k \geqslant 1$, and $\partial f_\phi/\partial x_i = (\partial f/\partial x_i) \circ I_\phi$ is strongly bounded on V for $i = 1,\dots,m$. It only remains to show that $\partial f_\phi/\partial x_{m+1}$ is strongly bounded on V. For $(t_1,\dots,t_{m+1}) \in V$ we have

$$\frac{\partial f_\phi}{\partial x_{m+1}}(t_1,\dots,t_m,t_{m+1}) \;=\; \phi'(t_{m+1}) \cdot \big(\frac{\partial f}{\partial x_{m+1}} \circ I_\phi\big)(t_1,\dots,t_m,t_{m+1}),$$

and by the properties of the map a we have for all $(t_1,\dots,t_m,t_{m+1}) \in V$,

$$\big|\big(\frac{\partial f}{\partial x_{m+1}} \circ I_\phi\big)(t_1,\dots,t_m,t_{m+1})\big| \;\leqslant\; 2\big|\frac{\partial f}{\partial x_{m+1}}\big(\gamma \circ \phi\big)(t_{m+1})\big|.$$

Combining the last two displays it is enough to strongly bound

$$\phi' \cdot \frac{\partial f}{\partial x_{m+1}} \circ (\gamma \circ \phi)$$

on $\pi_{m+1}(V)$. Since Φ is a k-reparametrization of $g|_{\pi_{m+1}(V)}$, we have:

(i) $(\gamma \circ \phi)'$ is strongly bounded on $\pi_{m+1}(V)$, and

(ii) $\big(f \circ \gamma \circ \phi\big)'$ is strongly bounded on $\pi_{m+1}(V)$.

Let $\gamma = (\gamma_1, \dots, \gamma_m, \gamma_{m+1})$, $\gamma_i : \pi_{m+1}(V_0) \to R$. By the Chain Rule (subsection "Differentiability" in Section 4.2), we have on $\pi_{m+1}(V)$:

$$\big(f \circ \gamma \circ \phi\big)' \;=\; \sum_{i=1}^{m} (\gamma_i \circ \phi)' \cdot \frac{\partial f}{\partial x_i} \circ (\gamma \circ \phi) \;+\; \phi' \cdot \frac{\partial f}{\partial x_{m+1}} \circ (\gamma \circ \phi)$$

Now $\partial f/\partial x_i$ for $i = 1, \dots, m$ is strongly bounded, so by (i) the above $\sum_{i=1}^{m}$ is strongly bounded on $\pi_{m+1}(V)$. Also the left hand side is strongly bounded on $\pi_{m+1}(V)$ by (ii), hence the remaining term $\phi' \cdot \frac{\partial f}{\partial x_{m+1}} \circ (\gamma \circ \phi)$ on the right is strongly bounded on $\pi_{m+1}(V)$ as well, which we already know to be enough. $\square$

Corollary 7.14. *Let $k, n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable C^1-map. Suppose also that $\partial f/\partial x_i$ is strongly bounded for $i = 1, \dots, m$. Then there is a k-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi/\partial x_i$ is strongly bounded on V for $i = 1, \dots, m+1$.*

Proof. For $n = 1$ this is Lemma 7.13. As an inductive assumption, let $f : U \to R^n$ be as in the hypothesis of the corollary and Φ and V as in its conclusion. Let $g : U \to R$ be a strongly bounded definable C^1-function such that $\partial g/\partial x_i$ is strongly bounded for $i = 1, \dots, m$. Then the strongly bounded definable C^1-map $(f, g) : U \to R^{n+1}$ has strongly bounded partial $\partial(f,g)/\partial x_i = (\partial f/\partial x_i, \partial g/\partial x_i)$ for $i = 1, \dots, m$. It now suffices to show that there is a k-parametrization Θ of a cofinite subset of $(0,1)$ and a set $W \Subset U$ such that for all $\theta \in \Theta$: $I_\theta(W) \subseteq U$, $(f,g)_\theta$ is of class C^1 on W, and $\partial(f,g)_\theta/\partial x_i$ is strongly bounded on W for $i = 1, \dots, m+1$. To construct Θ and W, let $\phi \in \Phi$. Then applying Lemma 7.13 to the function $g_\phi : V \to R$ gives a k-parametrization Ψ_ϕ of a cofinite subset of $(0,1)$ and a set $V_\phi \Subset V$ such that for all $\psi \in \Psi_\phi$: $I_\psi(V_\phi) \subseteq V$ and $(g_\phi)_\psi = g_{\phi \circ \psi}$ is of class C^1 on V_ϕ, and $\partial g_{\phi,\psi}/\partial x_i$ is strongly bounded on V_ϕ. Now we set

$$\Theta := \{\phi \circ \psi : \; \phi \in \Phi, \; \psi \in \Psi_\phi\}, \quad W := \bigcap_{\phi \in \Phi} V_\phi.$$

It follows easily from Lemma 7.4 that Θ and W have the desired properties. $\square$

To state the next corollary, let U be a definable open subset of R^{m+1}. Recall that for $t \in R$ we have the definable open subset U^t of R^m given by

$$U^t \;=\; \{(t_1, \dots, t_m) \in R^m : \; (t_1, \dots, t_m, t) \in U\}.$$

We call a definable map $f : U \to R^n$ *of class C^k in the first m variables* if for every $t \in R$ the (definable) map

$$f^t : U^t \to R^n, \qquad (t_1, \dots, t_m) \mapsto f(t_1, \dots, t_m, t)$$

is of class C^k. In that case $f^{(\alpha)}$ for $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ denotes the definable map

$$(t_1, \ldots, t_m, t) \mapsto (f^t)^{(\alpha)}(t_1, \ldots, t_m) \ : \ U \to R^n,$$

which for fixed t is continuous as a function of $(t_1, \ldots, t_m)$.

Corollary 7.15. *Let $k, n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable map that is of class C^k in the first m variables, such that $f^{(\alpha)}$ is strongly bounded for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$. Then for every $l \leqslant k$ there is a $V_l \Subset U$ and a k-parametrization Φ_l of a cofinite subset of $(0,1)$ such that for all $\phi \in \Phi_l$: $I_\phi(V_l) \subseteq U$, f_ϕ is of class C^k on V_l and $f_\phi^{(\alpha)} := \big(f_\phi\big)^{(\alpha)}$ is strongly bounded on V_l for all $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l$.*

Proof. The last sentence in the subsection on C^k-maps in Section 4.2 gives $V_0 \Subset U$ such that f is of class C^k on V_0. Then V_0 and $\Phi_0 = \{\mathrm{id}|_{(0,1)}\}$ have the desired properties for $l = 0$. Suppose, inductively, that $l < k$ and V_l and Φ_l are as stated in the Corollary. Let

$$\Delta \ := \ \{\alpha \in \mathbb{N}^{m+1} \ : \ |\alpha| \leqslant k-1, \ \alpha_{m+1} \leqslant l\},$$

set $\widetilde{n} := \#\Delta \cdot \#\Phi_l$, and let $F_1, \ldots, F_{\widetilde{n}} : V_l \to R^n$ enumerate the set of C^1-maps

$$\{f_\phi^{(\alpha)} : \ V_l \to R^n : \ \alpha \in \Delta, \ \phi \in \Phi_l\}.$$

Then we can apply Corollary 7.14 to $F := (F_1, \ldots, F_{\widetilde{n}}) : V_l \to R^{\widetilde{n}\cdot n}$ in the role of f, and V_l, $\widetilde{n} \cdot n$ instead of U, n. This gives a k-parametrization Ψ of a cofinite subset of $(0,1)$ and a set $V_{l+1} \Subset V_l$ such that for all $\psi \in \Psi$: $I_\psi(V_{l+1}) \subseteq V_l$, F_ψ is of class C^1 on V_{l+1}, and $\partial F_\psi / \partial x_i$ is strongly bounded on V_{l+1} for $i = 1, \ldots, m+1$. Next we set

$$\Phi_{l+1} \ := \ \{\phi \circ \psi : \ \phi \in \Phi_l, \ \psi \in \Psi\}.$$

Then Φ_{l+1} is a k-parametrization of a cofinite subset of $(0,1)$ and $I_\theta(V_{l+1}) \subseteq U$, with f_θ of class C^k for all $\theta \in \Phi_{l+1}$.

Let $\theta = \phi \circ \psi$ with $\phi \in \Phi_l, \psi \in \Psi$ and let $\alpha \in \mathbb{N}^{m+1}$, $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l+1$; it remains to show that then $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. If $\alpha_{m+1} = 0$, then this holds because $f_\theta^{(\alpha)} = \big(f_\phi^{(\alpha)}\big)_\psi$ and $f_\phi^{(\alpha)}$ is strongly bounded on V_l. Suppose that $\alpha_{m+1} > 0$. Then $\alpha = \beta + (0, \ldots, 0, j)$ with $\beta_{m+1} = 0$ and $j = \alpha_{m+1} \geqslant 1$, so for $a = (a_1, \ldots, a_m, a_{m+1}) \in V_{l+1}$ we have

$$\begin{aligned} f_\theta^{(\alpha)}(a) \ &= \ \frac{\partial^j f_\theta^{(\beta)}}{\partial x_{m+1}^j}(a) \ = \ \frac{\partial^j \big(f_\phi^{(\beta)}\big)_\psi}{\partial x_{m+1}^j}(a) \\ &= \ \sum_{i=1}^{j} \frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}\big(a_1, \ldots, a_m, \psi(a_{m+1})\big) \cdot p_{ij}\big(\psi^{(1)}(a_{m+1}), \ldots, \psi^{(j-i+1)}(a_{m+1})\big) \end{aligned}$$

using Lemma 7.3 and the polynomials p_{ij} from that lemma for the last equality. Since we assumed inductively that the $\frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}$ are strongly bounded on V_l and $\psi^{(1)}, \ldots, \psi^{(k)}$ are strongly bounded on $(0,1)$, $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. $\square$

7.4 Finishing the proofs of the parametrization theorems

In this section we assume that our ambient o-minimal field R is $\aleph_0$-saturated. We consider the following statements depending on m:

$(\mathrm{I})_m$ For all $k, n \geqslant 1$, every strongly bounded definable map $f : (0,1)^m \to R^n$ has a k-reparametrization.

$(\mathrm{II})_m$ For all $k \geqslant 1$, every strongly bounded definable set $X \subseteq R^{m+1}$ has a k-parametrization.

It is clear that $(\mathrm{I})_0$ and $(\mathrm{II})_0$ hold; $(\mathrm{I})_1$ holds by Corollary 7.10. We proceed by induction to show that $(\mathrm{I})_m$ and $(\mathrm{II})_m$ hold for all m. So let $m \geqslant 1$ and suppose that $(\mathrm{I})_l$ holds for all $l \leqslant m$ and that $(\mathrm{II})_l$ holds for all $l < m$. We show that then $(\mathrm{II})_m$ holds and next that $(\mathrm{I})_{m+1}$ holds. For $(\mathrm{II})_m$, let $k \geqslant 1$ and let $X \subseteq R^{m+1}$ be definable and strongly bounded. In order to show that X has a k-parametrization we can reduce to the case that X is a cell in R^{m+1}; we do the more difficult of the two cases, namely $X = (f,g)_Y$ where Y is a (strongly bounded) cell in R^m, and $f, g : Y \to R$ are strongly bounded continuous definable functions with $f(y) < g(y)$ for all $y \in Y$; the other case, where X is the graph of such a function $Y \to R$, is left to the reader.

Using $(\mathrm{II})_{m-1}$ we have a k-parametrization Φ of Y. Set $l := \dim Y$. Let $\phi \in \Phi$ be given. Then $\phi : (0,1)^l \to Y$ and $(\mathrm{I})_l$ gives a k-reparametrization Ψ_ϕ of the map $(f \circ \phi, g \circ \phi) : (0,1)^l \to R^2$. For $\psi \in \Psi_\phi$ we have $\psi : (0,1)^l \to (0,1)^l$, and we define $\theta_{\phi,\psi} : (0,1)^{l+1} \to X$ by

$$\theta_{\phi,\psi}(s,t) \ := \ \big((\phi \circ \psi)(s),\ (1-t) \cdot (f \circ \phi \circ \psi)(s) + t \cdot (g \circ \phi \circ \psi)(s)\big)$$

where $(s,t) = (s_1, \dots, s_l, t) \in (0,1)^{l+1}$. Then the set $\{\theta_{\phi,\psi} :\ \phi \in \Phi,\ \psi \in \Psi_\phi\}$ is readily seen to be a k-parametrization of X, and we have established $(\mathrm{II})_m$.

For $(\mathrm{I})_{m+1}$ we need only do the case $n = 1$ by the remark following the proof of Lemma 7.9. So let $k \geqslant 1$ and let $f : (0,1)^{m+1} \to R$ be a strongly bounded definable function; our job is to show that f has a k-reparametrization.

In the rest of this proof t ranges over the interval $(0,1)$. By $(\mathrm{I})_m$ there is for all t a k-reparametrization of the function $f^t : (0,1)^m \to R$ given by $f^t(s) = f(s,t)$. Now R is $\aleph_0$-saturated, and together with Definable Selection (see end of Section 4.3 for details on this use of model-theoretic compactness) this yields an $N \in \mathbb{N}^{\geqslant 1}$ and definable families $(\phi_1^t), \dots, (\phi_N^t)$ of maps

$$\phi_j^t \ : \ (0,1)^m \to (0,1)^m \qquad (j = 1, \dots, N)$$

such that $\Phi^t := \{\phi_1^t, \dots, \phi_N^t\}$ is for every t a k-reparametrization of f^t.

Now, for $j = 1, \dots, N$ we define the function $f_j : (0,1)^{m+1} \to R$ by

$$f_j(s,t) \ := \ f\big(\phi_j(s,t), t\big),$$

where $\phi_j : (0,1)^{m+1} \to (0,1)^m$ is given by $\phi_j(s,t) := \phi_j^t(s)$. Consider the map

$$F \ := \ \big(\phi_1, \dots, \phi_N, f_1, \dots, f_N\big) \ : \ (0,1)^{m+1} \to R^{Nm+N}.$$

Then the hypotheses of Corollary 7.15 are satisfied for F and $(0,1)^{m+1}$ in the role of f and U, and $Nm + N$ for n: this is just restating that Φ^t is a k-reparametrization of f^t, uniformly in t. The conclusion of that corollary for $l = k$ gives a set $V \Subset (0,1)^{m+1}$ and a k-parametrization Ψ of a cofinite subset of $(0,1)$ such that for all $\psi \in \Psi$ the map $F_\psi : (0,1)^{m+1} \to R^{Nm+N}$ is of class C^k on V with strongly bounded $F_\psi^{(\alpha)}$ on V for all $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| \leqslant k$.

For $j = 1, \dots, N$ and $\psi \in \Psi$, let $\phi_j * \psi : (0,1)^{m+1} \to (0,1)^{m+1}$ be given by

$$(\phi_j * \psi)(s,t) \ := \ \big(\phi_j(s, \psi(t)), \psi(t)\big) \ = \ \big(\phi_j^{\psi(t)}(s), \psi(t)\big).$$

The images of the $\psi \in \Psi$ cover a set $(0,1) \setminus \{t_1, \dots, t_d\}$ and for every t the images of $\phi_1^t, \dots, \phi_N^t$ cover $(0,1)^m$, and thus the images of the above $\phi_j * \psi$ cover $(0,1)^{m+1}$ apart from finitely many hyperplanes $x_{m+1} = t_i$. Setting

$$W \ := \bigcup_{1 \leqslant j \leqslant N,\ \psi \in \Psi} (\phi_j * \psi)(V)$$

it follows that the definable set $(0,1)^{m+1} \setminus W$ has dimension $\leqslant m$. Using the now established $(\mathrm{II})_m$, let Θ_1 be a k-parametrization of V and Θ_2 a k-parametrization of $(0,1)^{m+1} \setminus W$. For $\theta \in \Theta_2$ we have $\theta : (0,1)^l \to (0,1)^{m+1}$ with $l \leqslant m$ and then $(\mathrm{I})_l$ yields a k-reparametrization Λ_θ of the function $f \circ \theta : (0,1)^l \to R$. The required k-reparametrization of f is now given by

$$\{(\phi_j * \psi) \circ \chi : \ j = 1, \dots, N, \ \psi \in \Psi, \chi \in \Theta_1\} \cup \{\theta \circ \hat{\lambda} : \ \theta \in \Theta_2, \lambda \in \Lambda_\theta\}$$

where $\hat{\lambda} : (0,1)^{m+1} \to (0,1)^l$ (for $l \leqslant m$ as above) is given by $\hat{\lambda}(t_1, \dots, t_{m+1}) := \lambda(t_1, \dots, t_l)$. This finishes the proof of $(\mathrm{I})_{m+1}$, and the induction is complete. In particular, Theorem 7.1 is now established. Theorem 7.2 requires one more easy step and we leave this to the reader.

Corollary 7.16. *Let $k, n \geqslant 1$; suppose $X \subseteq [-1,1]^n$ is definable, $d := \dim X \geqslant 0$. Then there exists a finite set Φ of definable C^k-maps $f : (0,1)^d \to R^n$ such that*

(i) $\bigcup_{f \in \Phi} \mathrm{image}(f) = X$;

(ii) $|f^{(\alpha)}(t)| \leqslant 1$ *for all* $f \in \Phi$ *and* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$ *and all* $t \in (0,1)^d$.

Proof. Let Φ^* be a k-parametrization of X. Then (i) holds for Φ^* instead of Φ and (ii) holds for Φ^* instead of Φ, with a certain $c \in \mathbb{N}^{\geqslant 1}$ in place of 1. Cover $(0,1)^d$ with $(c+1)^d$ translates of the 'box' $(0, \frac{1}{c})^d$ and for each such translate B, let $\lambda_B : (0,1)^d \to B$ be the obvious affine bijection. Then the set of maps $f \circ \lambda_B$ as f varies over Φ^* and B over the above translates is the required Φ, since $(f \circ \lambda_B)^{(\alpha)} = c^{-|\alpha|} \cdot \big(f^{(\alpha)} \circ \lambda_B\big)$ for such f and B and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. $\square$

Definable Selection and $\aleph_0$-saturation lead to a uniform version, as explained in more detail at the end of Section 4.3:

Corollary 7.17. *Let d, k, m, n be given with $k, n \geqslant 1$ and suppose $E \subseteq R^m$ and*

$$Z \ \subseteq \ E \times [-1,1]^n \ \subseteq \ R^{m+n}$$

are definable with $\dim Z(s) = d$ *for all $s \in E$. Then there are $N \in \mathbb{N}^{\geqslant 1}$ and a definable set $F \subseteq E \times R^d \times R^{Nn}$ such that for all $s \in E$, $F(s) \subseteq R^d \times R^{Nn}$ is the graph of a C^k-map $(f_1, \dots, f_N) : (0,1)^d \to (R^n)^N = R^{Nn}$ such that:*

(i) $\bigcup_{j=1}^N \mathrm{image}(f_j) = Z(s)$;

(ii) $|f_j^{(\alpha)}(t)| \leqslant 1$ *for* $j = 1, \dots, N$, $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$, *and* $t \in (0,1)^d$.

The proof of Corollary 7.17 uses that R is $\aleph_0$-saturated, but this corollary goes through without this assumption: pass to an $\aleph_0$-saturated elementary extension and then go back. Thus it applies to o-minimal expansions of the real field to give Theorem 5.3, and we can also combine it with Theorem 6.6 to give:

Corollary 7.18. *Let $n \geqslant 1$ and let an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field be given. Suppose $E \subseteq \mathbb{R}^m$ and $Z \subseteq E \times [-1,1]^n \subseteq \mathbb{R}^{m+n}$ are definable. Then there is for every $\varepsilon > 0$ an $e = e(\varepsilon, n)$ and a K with the following property: for all $s \in E$ with $\dim Z(s) < n$ and all T, at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$.*

The expression "$e = e(\varepsilon, n)$" means: e can be chosen to depend only on ε and n. The proof below uses the numbers $\varepsilon(d,n,e) := \frac{dneD(n,e)}{B(d,n,e)}$ from Section 6.1.

Proof. Replacing E by finitely many definable subsets over each of which $\dim Z(s)$ takes a given value, we arrange that for a certain $d < n$ we have $\dim Z(s) = d$ for all $s \in E$. If $d = 0$, then we have $K \in \mathbb{N}^{\geqslant 1}$ such that $\#Z(s) \leqslant K$ for all $s \in E$, and so at most K hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant 1$ are enough to cover $Z(s)$. Assume $d \geqslant 1$. Take $e \geqslant 1$ such that $\varepsilon(d,n,e) \leqslant \varepsilon$ and set $k := b(d,n,e) + 1$ as in Theorem 6.6. Corollary 7.17 gives an $N \in N^{\geqslant 1}$ and for every $s \in E$ maps $f_1, \dots, f_N : (0,1)^d \to R^n$ of class C^k such that $Z(s) = \bigcup_{j=1}^N \operatorname{image}(f_j)$ and $|f_j^{(\alpha)}(t)| \leqslant 1$ for $j = 1, \dots, N$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ and all $t \in (0,1)^d$. Applying Theorem 6.6 to each map f_j separately we obtain that for $K := N \cdot C(d,n,e)$ at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$. $\square$

CHAPTER 8

Strengthening and Extending the Counting Theorem

In this chapter we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, and *definable* is with respect to $\widetilde{\mathbb{R}}$. Throughout $n \geqslant 1$ and $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

A closer look at the proof of Theorem 5.8 gives useful extra information about the definable subsets $V(s)$ of $X(s)^{\mathrm{alg}}$: Theorem 8.4. To express this information efficiently requires the notion of a *block family*, which is here simpler than in [51] and well suited to the inductive set-up of Section 5.2. See the subsection *Dimension* in Section 4.2 for the local dimension $\dim_a$ used in defining blocks.

8.1 A block family version

Let $d \leqslant n$. A *block in* $\mathbb{R}^n$ *of dimension* d is a definable connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$ for which $\dim_a A = d$ for all $a \in A$. Thus the empty subset of $\mathbb{R}^n$ counts as a block in $\mathbb{R}^n$ of dimension d, but if B is a nonempty block in $\mathbb{R}^n$ of dimension d, then $\dim B = d$. Also, a nonempty block of dimension 0 in $\mathbb{R}^n$ consists just of one point. A *block family in* $\mathbb{R}^n$ *of dimension* d is a definable set $V \subseteq E \times \mathbb{R}^n$ all whose sections $V(s)$ are blocks in $\mathbb{R}^n$ of dimension d. Here are two easy lemmas:

Lemma 8.1. *Suppose $U \subseteq \mathbb{R}^m$ is open and semialgebraic, $m \geqslant 1$, and $f : U \to \mathbb{R}^n$ is semialgebraic and maps U homeomorphically onto $f(U)$. Then f maps any block $B \subseteq U$ in $\mathbb{R}^m$ of dimension $d \leqslant m$ onto a block $f(B)$ in $\mathbb{R}^n$ of dimension d.*

In the proof of Theorem 8.4 we apply Lemma 8.1 for every $I \subseteq \{1, \ldots, n\}$ to the map $a \mapsto b : \{a \in \mathbb{R}^n : a_i \neq 0 \text{ for } i \in I\} \to \mathbb{R}^n$ with $b_i = a_i^{-1}$ for $i \in I$ and $b_i = a_i$ for $i \notin I$; these maps extend the maps f_I from Section 5.2.

Lemma 8.2. *Let B be a block in $\mathbb{R}^n$ of dimension $d \leqslant n$. Then B is a union of connected semialgebraic subsets of dimension d.*

Proof. Take semialgebraic $A \subseteq \mathbb{R}^n$ such that $\dim_a A = d$ for all $a \in A$, and B is an open subset of A. For $b \in B$, take a semialgebraic open neighborhood U of b in A such that $U \subseteq B$. Now use that the connected components of U are open in A, by Corollary 4.3, and thus of dimension d. $\square$

Corollary 8.3. *Let $Y \subseteq \mathbb{R}^n$ and $1 \leqslant d \leqslant n$.*

(i) *if $B \subseteq Y$ and B is a block in $\mathbb{R}^n$ of dimension d, then $B \subseteq Y^{\mathrm{alg}}$;*

(ii) *if V is a block family in $\mathbb{R}^n$ of dimension d, then the union of the sections of V that are contained in Y is contained in Y^{alg}.*

For the inductive proof below we also define a *block family in* $\mathbb{R}^0$ *of dimension* 0 to be a definable set $V \subseteq E \times \mathbb{R}^0$, with $E \times \mathbb{R}^0$ identified with E in the obvious way.

Theorem 8.4. *Let ε be given. Then there are a natural number $N = N(X, \varepsilon) \geqslant 1$, a block family $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ in $\mathbb{R}^n$ of dimension $d_j \leqslant n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \ldots, N$, and a constant $c = c(X, \varepsilon)$, such that:*

(i) *$V_j(s,t) \subseteq X(s)$ for $j = 1, \ldots, N$ and $(s,t) \in E \times F_j$;*

(ii) *for all T and all $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most cT^{ε} blocks $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.*

This yields an improved Theorem 5.8 as follows. Let $V_1, \ldots, V_N$ and c be as in Theorem 8.4. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \ := \bigcup_{d_j \geqslant 1, t \in F_j} V_j(s,t)$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}(X(s) \setminus V(s), T) \leqslant cT^{\varepsilon}$ for all T.

Proof. If Theorem 8.4 holds for definable sets $X_1, \ldots, X_\nu \subseteq E \times \mathbb{R}^n$, $\nu \in \mathbb{N}$, then also for $X = X_1 \cup \cdots \cup X_\nu$. We shall tacitly use this below.

We proceed by induction on n, and follow the proof of Theorem 5.8 closely. Set $V_0(s) :=$ interior of $X(s)$. Then Theorem 4.2 and Proposition 4.4 give $M \in \mathbb{N}^{\geqslant 1}$ such that for all $s \in E$,

$$\#\{\text{connected components of } V_0(s)\} \ \leqslant \ M.$$

Definable Selection (Proposition 4.10) and the lexicographic ordering on $\mathbb{R}^n$ give definable subsets $V_1, \ldots, V_M$ of $E \times \mathbb{R}^n$ such that for all $s \in E$ the sets $V_1(s), \ldots, V_M(s)$ are connected (possibly empty), open in $V_0(s)$, pairwise disjoint, with $V(s) \ = \ \bigcup_{i=1}^{M} V_i(s)$. So $V_1, \ldots, V_M$ are block families in $\mathbb{R}^n$ of dimension n; we make them the first M of the $V_1, \ldots, V_N$ to be constructed. Now replacing X with $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. Applying Lemma 8.1 to the natural extensions of the maps f_I, $I \subseteq \{1, \ldots, n\}$, we arrange also that $X(s) \subseteq [-1,1]^n$ for all $s \in E$.

Next, take e and $k = k(n,e)$ as in the proof of Theorem 5.7. So we have $C = C(X, \varepsilon) \in \mathbb{R}^{>}$ such that for any $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most $CT^{\varepsilon/2}$ many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$. Therefore it suffices to find $V_1, \ldots, V_N$ and c as in the theorem but with (ii) replaced by

(ii)* for all T, all $s \in E$, and all hypersurfaces H of degree $\leqslant e$, $(X(s) \cap H)(\mathbb{Q}, T)$ is covered by at most $\frac{c}{C} T^{\varepsilon/2}$ blocks $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$;

We use again the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \ldots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$, and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$, $l = 1, \ldots, L$, as in the proof of Theorem 5.7. Since $n_l < n$, the induction assumption gives a natural number $N_l = N(Y_l, \varepsilon) \geqslant 1$, a block family

$$W_{l,i} \ \subseteq \ \big((E \times F) \times G_{l,i}\big) \times \mathbb{R}^{n_l}$$

in $\mathbb{R}^{n_l}$ of dimension $d_{l,i} \leqslant n_l$ with definable $G_{l,i} \subseteq \mathbb{R}^{m_{l,i}}$, for $i = 1, \ldots, N_l$, and $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^{>}$, such that (i)′ $W_{l,i}(s,t,g) \subseteq Y_l(s,t)$ for $i = 1, \dots, N_l$, $(s,t,g) \in (E \times F) \times G_{l,i}$;

(ii)′ for all T and all $(s,t) \in E \times F$, $Y_l(s,t)(\mathbb{Q}, T)$ is covered by at most $B_l T^{\varepsilon/2}$ blocks $W_{l,i}(s,t,g)$, $(1 \leqslant i \leqslant N_l,\ g \in G_{l,i})$.

Set $N := N_1 + \cdots + N_L$, and for $l = 1, \dots, L$, $1 \leqslant i \leqslant N_l$ and $j = N_1 + \cdots + N_{l-1} + i$, set $F_j := F \times G_{l,i}$, and let $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ be the definable set given by

$$V_j\big(s, (t,g)\big) = \mathcal{C}_l(t) \cap p_{\boldsymbol{i}l}^{-1}\big(W_{l,i}(s,t,g)\big), \qquad (s \in E,\ t \in F,\ g \in G_{l,i}),$$

so V_j is a block family in $\mathbb{R}^n$ of dimension $d_{l,i} < n$, by Lemma 8.1. It is easy to check that $V_1, \dots, V_N$ and $c := C(B_1 + \cdots + B_L)$ are as desired. □

8.2 Generalizations

Counting points over $\mathbb{Q}$-linear subspaces of $\mathbb{R}$. In this subsection we fix $d \geqslant 1$. Instead of rational points we now allow points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ of dimension $\leqslant d$. Let $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$, and set $\mathbb{Q}\lambda := \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_d \subseteq \mathbb{R}$. For $a \in \mathbb{Q}\lambda$ we set

$$\mathrm{H}_\lambda(a) \ := \ \min\{\mathrm{H}(q) :\ q \in \mathbb{Q}^d,\ q \cdot \lambda = a\} \in \mathbb{N}^{\geqslant 1}.$$

Here $q \cdot \lambda := q_1\lambda_1 + \cdots + q_d\lambda_d$. We define a height function H_λ on $(\mathbb{Q}\lambda)^n \subseteq \mathbb{R}^n$ by

$$\mathrm{H}_\lambda(a) \ = \ \max\{\mathrm{H}_\lambda(a_1), \dots, \mathrm{H}_\lambda(a_n)\} \ \text{ for } a = (a_1, \dots, a_n) \in (\mathbb{Q}\lambda)^n.$$

For $Y \subseteq \mathbb{R}^n$ we introduce its finite subsets $Y_\lambda(T)$ and their cardinalities:

$$Y_\lambda(T) \ := \ \{a \in Y \cap (\mathbb{Q}\lambda)^n :\ \mathrm{H}_\lambda(a) \leqslant T\}, \qquad \mathrm{N}_\lambda(Y,T) \ := \ \#Y_\lambda(T).$$

Theorem 8.5. *Let any definable $Y \subseteq \mathbb{R}^n$ and any ε be given. Then there is a constant $c = c(Y, d, \varepsilon) \in \mathbb{R}^>$ such that for all T and all $\lambda \in \mathbb{R}^d$,*

$$\mathrm{N}_\lambda(Y^{\mathrm{tr}}, T) \ \leqslant \ cT^\varepsilon.$$

Proof of Theorem 8.5. First a useful lemma about blocks:

Lemma 8.6. *If B is a block in $\mathbb{R}^n$ (of some dimension) and $p, q \in B$, then $\gamma(0) = p$ and $\gamma(1) = q$ for some continuous semialgebraic path $\gamma : [0,1] \to B$.*

Proof. Even better, let B be a connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$, and let $p \in B$. We claim: there is for every $q \in B$ a continuous semialgebraic path $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma([0,1]) \subseteq B$. To see this, let $B(p)$ be the set of all $q \in B$ for which there is such a path. The sets $B(p)$ as p ranges over B form a partition of B, so it is enough to show that the $B(p)$ are open in B, which reduces to showing that $B(p)$ is a neighborhood of p in B. Now B is open in A, so we have a semialgebraic open subset U of A with $p \in U \subseteq B$. The connected component C of U with $p \in C$ is open in U and semialgebraic by Corollary 4.3 and the remarks preceding it. These remarks also give $C \subseteq B(p)$. □

Corollary 8.7. *If B is a block in $\mathbb{R}^m$ (of some dimension), A is a semialgebraic subset of $\mathbb{R}^m$ with $B \subseteq A$, and $\phi : A \to \mathbb{R}^n$ is a continuous semialgebraic map such that $\phi(B)$ has more than one point, then $\phi(B) = \phi(B)^{\mathrm{alg}}$.*

Proof. Use that the ϕ-image of a path γ as in Lemma 8.6 is a connected semialgebraic subset of $\phi(B)$. $\square$

The next result is basically a consequence of Theorem 8.4, as the proof will show.

Theorem 8.8. *Given ε, there are a natural number $N = N(X, d, \varepsilon) \geqslant 1$, a definable set $V_j \subseteq (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X, d, \varepsilon)$, such that for $j = 1, \dots, N$ and all $(s, \lambda, t) \in E \times \mathbb{R}^d \times F_j$:*

(i) *$V_j(s, \lambda, t) \subseteq X(s)$ and $V_j(s, \lambda, t)$ is connected;*

(ii) *if $\dim V_j(s, \lambda, t) \geqslant 1$, then $V_j(s, \lambda, t) \subseteq X(s)^{\mathrm{alg}}$,*

and such that for all T and $(s, \lambda) \in E \times \mathbb{R}^d$, the set $X(s)_\lambda(T)$ is covered by at most cT^ε sections $V_j(s, \lambda, t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.

This yields a family version of Theorem 8.5 as follows. Let $V_1, \dots, V_N$ and c be as in Theorem 8.8. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \ := \ \bigcup \{V_j(s, \lambda, t) : \ 1 \leqslant j \leqslant N,\ (\lambda, t) \in \mathbb{R}^d \times F_j,\ \dim V_j(s, \lambda, t) \geqslant 1\}$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}_\lambda(X(s) \setminus V(s), T) \leqslant cT^\varepsilon$ for all T.

Proof. Let $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^n \to \mathbb{R}^n$ be given by $\pi(\lambda, a_1, \dots, a_n) = (\lambda \cdot a_1, \dots, \lambda \cdot a_n)$, where $a_1, \dots, a_n \in \mathbb{R}^d$. Set

$$X^* \ := \ \{(s, \lambda, a_1, \dots, a_n) \in (E \times \mathbb{R}^d) \times (\mathbb{R}^d)^n : \ \big(s, \pi(\lambda, a_1, \dots, a_n)\big) \in X\},$$

viewed as a definable family of subsets of $(\mathbb{R}^d)^n$. Note that for $s \in E$ and $\lambda \in \mathbb{R}^d$,

$$(*) \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)\big) \ \subseteq \ X(s), \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)(\mathbb{Q}, T)\big) \ = \ X(s)_\lambda(T).$$

We apply Theorem 8.4 to X^* in the role of X. It gives $N = N(X^*, \varepsilon) \geqslant 1$, a block family $V_j^* \subseteq (E \times \mathbb{R}^d \times F_j) \times (\mathbb{R}^d)^n$ in $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X^*, \varepsilon)$ such that:

(i)* $V_j^*(s, \lambda, t) \subseteq X^*(s, \lambda)$ for $j = 1, \dots, N$ and (s, λ, t) in $E \times \mathbb{R}^d \times F_j$;

(ii)* for all T and all $(s, \lambda) \in E \times \mathbb{R}^d$, the set $X^*(s, \lambda)(\mathbb{Q}, T)$ is covered by at most cT^ε sections $V_j^*(s, \lambda, t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.

Now we set for $j = 1, \dots, N$,

$$V_j \ := \ \{\big(s, \lambda, t, \pi(\lambda, a)\big) \in (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n : \ (s, \lambda, t, a) \in V_j^*\},$$

so $V_j(s, \lambda, t) = \pi\big(\{\lambda\} \times V_j^*(s, \lambda, t)\big)$ for $(s, \lambda, t) \in E \times \mathbb{R}^d \times F_j$. We now show that $V_1, \dots, V_N$ and $c(X, d, \varepsilon) := c(X^*, \varepsilon)$ have the desired properties. Clause (i) is satisfied using (i)* and $(*)$, and (ii) is satisfied in view of Corollary 8.7. The rest follows from (ii)* and $(*)$. $\square$

Extending the Counting Theorem to Algebraic Points. Throughout this subsection we fix $d \geqslant 1$. Instead of rational points we now count algebraic points whose coordinates are of degree at most d over $\mathbb{Q}$. We define the corresponding height of an algebraic number $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant d$ by

$$\mathrm{H}_d^{\mathrm{poly}}(\alpha) \ := \ \min\{\mathrm{H}(\xi) : \ \xi \in \mathbb{Q}^d, \ \alpha^d + \xi_1\alpha^{d-1} + \cdots + \xi_d = 0\} \in \mathbb{N}^{\geqslant 1}.$$

(For us this height is notationally more convenient than the height for real algebraic numbers used by Pila in [P2]. The two heights are related as follows, where we use an extra subscript P for the height in [P2]: for $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant d$,

$$\mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha) \ \leqslant \ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \ \leqslant \ \mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha)^2.$$

Thus the results below for our height also hold for the other height.)

We extend the above height to all $\alpha \in \mathbb{R}$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \infty$ if $[\mathbb{Q}(\alpha):\mathbb{Q}] > d$, and to all points $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \max\{\mathrm{H}_d^{\mathrm{poly}}(\alpha_1), \dots, \mathrm{H}_d^{\mathrm{poly}}(\alpha_n)\}$. For $Y \subseteq \mathbb{R}^n$ we introduce its finite subsets $Y_d(T)$ and their cardinalities:

$$Y_d(T) \ := \ \{\alpha \in Y : \ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \leqslant T\}, \qquad \mathrm{N}_d(Y,T) \ := \ \#Y_d(T).$$

Theorem 8.9. *Let $Y \subseteq \mathbb{R}^n$ be definable, and let ε be given. Then there is a constant $c = c(Y,d,\varepsilon)$ such that for all T,*

$$N_d(Y^{\mathrm{tr}},T) \ \leqslant \ cT^{\varepsilon}.$$

We shall use the following easy consequence of semialgebraic cell decomposition:

Lemma 8.10. *Let $A_{n,d} \subseteq \mathbb{R}^{n\times d} \times \mathbb{R}^n$ be the semialgebraic set*

$$\{(\xi,\alpha) \in \mathbb{R}^{n\times d} \times \mathbb{R}^n : \ \alpha_i^d + \xi_{i1}\alpha_i^{d-1} + \cdots + \xi_{id} = 0 \textit{ for } i = 1, \dots, n\}.$$

Then we have a natural number $L = L(n,d) \geqslant 1$, a semialgebraic set $D_l \subseteq \mathbb{R}^{n\times d}$ with a semialgebraic continuous map $\phi_l : D_l \to \mathbb{R}^n$, for $l = 1, \dots, L$, such that $A_{n,d} = \bigcup_{l=1}^{L} \mathrm{graph}(\phi_l)$. It follows that for all $\alpha \in \mathbb{R}^n$ with $\mathrm{H}_d^{\mathrm{poly}}(\alpha) < \infty$ there is an $l \in \{1, \dots, L\}$ and a $\xi \in D_l$ such that $\phi_l(\xi) = \alpha$ and $\mathrm{H}(\xi) = \mathrm{H}_d^{\mathrm{poly}}(\alpha)$.

Towards Theorem 8.9 we first prove something stronger:

Theorem 8.11. *Let ε be given. Then there are $N = N(X,d,\varepsilon) \in \mathbb{N}^{\geqslant 1}$, a definable set $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X,d,\varepsilon)$, such that for $j = 1, \dots, N$ and all $(s,t) \in E \times F_j$:*

(i) *$V_j(s,t) \subseteq X(s)$ and $V_j(s,t)$ is connected;*

(ii) *if $\dim V_j(s,t) \geqslant 1$, then $V_j(s,t) \subseteq X(s)^{\mathrm{alg}}$,*

and such that for all T and $s \in E$, the set $X(s)_d(T)$ is covered by at most cT^{ε} sections $V_j(s,t)$, $(1 \leqslant j \leqslant N, \ t \in F_j)$.

Proof. Let $\pi : \mathbb{R}^{n\times d} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be the obvious projection map. Take L and $\phi_1 : D_1 \to \mathbb{R}^n, \dots, \phi_L : D_L \to \mathbb{R}^n$ as in Lemma 8.10. Let $l \in \{1, \dots, L\}$. We set

$$\begin{aligned} X_l \ &:= \ \{(s,\xi,\alpha) \in E \times D_l \times \mathbb{R}^n : \ \alpha \in X(s), \ \phi_l(\xi) = \alpha\}, \\ Y_l \ &:= \ \{(s,\xi) \in E \times D_l : \ \xi \in \pi\big(X_l(s)\big)\} \ = \ \{(s,\xi) \in E \times D_l : \ \phi_l(\xi) \in X(s)\}, \end{aligned}$$

so for $s \in E$ we have $\phi_l\big(Y_l(s)\big) \subseteq X(s)$, and by Lemma 8.10, for all T,

$$X(s)_d(T) \;=\; \bigcup_{l=1}^{L} \phi_l\big(Y_l(s)(\mathbb{Q},T)\big).$$

We now apply Theorem 8.4 to Y_l in the role of X, and get $N_l = N_l(Y_l,\varepsilon) \in \mathbb{N}^{\geqslant 1}$, a block family $V_{l,i} \subseteq (E \times F_{l,i}) \times \mathbb{R}^{n\times d}$ in $\mathbb{R}^{n\times d}$ with definable $F_{l,i} \subseteq \mathbb{R}^{m_{l,i}}$, for $i = 1,\dots,N_l$, and a constant $c_l = c_l(Y_l,\varepsilon) \in \mathbb{R}^{>}$ such that:

(i) $V_{l,i}(s,t) \subseteq Y_l(s)$ for $i = 1,\dots,N_l$ and (s,t) in $E \times F_{l,i}$;

(ii) for all T and all $s \in E$, the set $Y_l(s)(\mathbb{Q},T)$ is covered by at most $c_l T^{\varepsilon}$ blocks $V_{l,i}(s,t)$, ($1 \leqslant i \leqslant N_l$, $t \in F_{l,i}$).

Set $N := N_1+\cdots+N_L$, and for $1 \leqslant i \leqslant N_l$ and $j = N_1+\cdots+N_{l-1}+i$, set $F_j := F_{l,i}$, and let $V_j \subseteq (E\times F_j)\times\mathbb{R}^n$ be the definable set given by

$$V_j(s,t) \;:=\; \phi_l\big(V_{l,i}(s,t)\big), \qquad (s \in E,\ t \in F_j).$$

It is easily verified using Lemma 8.7 that $V_1,\dots,V_N$ and $c(X,d,\varepsilon) := c_1 + \cdots + c_L$ have the properties stated in the Theorem. $\square$

Just as with Theorem 8.8 this leads to a family version of Theorem 8.9 as follows. Let $V_1,\dots,V_N$ and c be as in Theorem 8.11. Take the definable set $V \subseteq E \times \mathbb{R}^n$ such that for all $s \in E$,

$$V(s) \;:=\; \bigcup\{V_j(s,t):\ 1 \leqslant j \leqslant N,\ t \in F_j,\ \dim V_j(s,t) \geqslant 1\}.$$

Then for all $s \in E$ and all T we have

$$V(s) \;\subseteq\; X(s)^{\mathrm{alg}} \quad \text{and} \quad \mathrm{N}_d(X(s)\setminus V(s),T) \;\leqslant\; cT^{\varepsilon}.$$

Part III

Analytic Ax-Kochen-Ersov theory including induced structure on coefficient field and monomial group

CHAPTER 9

The setup

In this part of the thesis we develop a extension theory for valued fields with *analytic structure* in parallel to the original theory of valued fields, which in addition to AKE-type results for these structures, leads to induced structure results for the coefficient field and monomial group.

Chapter 9 begins with an overview of our results, and then comprises a brief section on henselianity, and a section on ultranormed rings and restricted power series over them, including the Weierstrass theorems.

At the beginning of Section 10.1, we define for any complete ultranormed ring A subject to mild conditions the notion of *A-analytic ring*: each n-variable restricted power series over A yields an n-ary operation on any A-analytic ring. Starting in Section 10.2 we specialize to the case that A is a noetherian ring with an ideal $\mathcal{o}(A) \neq A$, such that $\bigcap_n \mathcal{o}(A)^n = \{0\}$ and A is $\mathcal{o}(A)$-adically complete. In Section 10.3 we define A-analytic valuation rings and in Section 10.4 we consider immediate extensions.

In Section 11.1 we introduce affinoid sets, and in Section 11.2 we apply Weierstrass preparation and division to investigate the structure of affinoid algebras. This allows us to complete the full array of extension results in Section 12.1. In Section 12.2 we end with an analytic AKE-type equivalence theorem, and use it to prove an induced structure result for coefficient fields and monomial groups.

9.1 An overview of the program

In the 1960s Ax and Kochen [3, 4, 5], and Ersov [33, 34, 35, 36] independently, developed a model theory for henselian valuation rings with significant applications to p-adic number theory. Since then there have been many generalizations and refinements, and AKE-theory remains a very active area of research. For example, in the 1980s Denef and van den Dries [23, 27] saw how to handle the ring of p-adic integers with analytic structure given by (restricted) power series. This led to the solution of a problem posed by Serre [59], and to a theory of p-adic subanalytic sets. Using "mixed" power series this was extended to a theory of rigid subanalytic sets over other henselian valuation rings with more complicated analytic structure by L. Lipshitz, Z. Robinson, R. Cluckers, and others, see [44, 45, 20, 21].

An interesting part of the original AKE-theory has so far not been extended to this analytic setting: in the equal characteristic 0 case one can add a predicate for a *coefficient field* (a lift of the residue field to the ambient field), and then the structure induced on this coefficient field can be shown to be just its pure field structure; likewise for a *monomial group*, that is, a lift of the value group.

In the analytic setting, there is only a partial result in this direction by Binyamini, Cluckers and Novikov [12, Proposition 2], and the usual approaches to analytic AKE-theory—based on direct reductions to ordinary AKE-theory by Weierstrass division "with parameters"—cannot be adapted to cover fully the induced structure aspect, as far as we know.

Here we do obtain the expected induced structure results in an analytic setting by taking another route, based on developing a theory of analytic valuation rings in closer analogy with ordinary valuation theory. Weierstrass division is still key, as in [23, 27], but now in a different way. Some of our work consists in generalizing a substantial part of [30] (with simpler basic notions and better notation), but we require also new ideas and tricks, in particular in Sections 10.3, 10.4, 12.1. We also remedy what seems to be a gap in [30].

Much of our analytic valuation theory is characteristic-free, but for the final analytic AKE-results in Section 12.2 we restrict to equicharacteristic 0 with the value group a $\mathbb{Z}$-group. We believe that the equicharacteristic 0 assumption there can be replaced by "finitely ramified mixed characteristic," but will leave this for another occasion. As to removing the $\mathbb{Z}$-group assumption, it is plausible one can do this using Lipshitz's mixed power series; we have not pursued this.

More on induced structure. Here we state in detail a typical case of our result on induced structure. First we say what it is in the classical (non-analytic) setting. Let C be a (coefficient) field. This yields the valuation ring $C[[t]]$ of formal power series in one variable t over C. We now expand the ring $C[[t]]$ to the structure $\big(C[[t]], C\big)$: a ring with a distinguished subset. Then a classical "induced structure" result is that if $\operatorname{char} C = 0$, any set $X \subseteq C^n$ which is definable in $\big(C[[t]], C\big)$ is even definable in the field C. (This can be proved along familiar lines, so we consider it as folklore knowledge, though we do not know an explicit reference. It seems this is still open for $\operatorname{char} C > 0$.) Here and below, n ranges over $\mathbb{N} = \{0, 1, 2, \dots\}$ and "definable" means "definable with parameters from the ambient structure".

We now equip $C[[t]]$ with analytic structure as follows: for each n we have the (Tate) ring $A\langle Y_1, \dots, Y_n\rangle$ of restricted power series in the distinct indeterminates $Y_1, \dots, Y_n$ over $A = C[[t]]$: it consists of the formal power series

$$f \;=\; f(Y_1, \dots, Y_n) \;=\; \sum_{\nu} a_\nu Y_1^{\nu_1} \cdots Y^{\nu_n}, \qquad \nu = (\nu_1, \dots, \nu_n) \text{ ranging over } \ \mathbb{N}^n,$$

with all $a_\nu \in A$ such that $a_\nu \to 0$, t-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$. Each such f gives rise to an n-ary operation on $C[[t]]$, namely

$$y \;=\; (y_1, \dots, y_n) \mapsto f(y_1, \dots, y_n) \;:\; C[[t]]^n \to C[[t]].$$

We expand the ring $C[[t]]$ to $C[[t]]_{\text{an}}$ by taking each such f as a new n-ary function symbol that names the above n-ary operation on $C[[t]]$. Further expansion yields the structure $\big(C[[t]]_{\text{an}}, C\big)$, and now our new induced structure result says that any set $X \subseteq C^n$ which is definable in $\big(C[[t]]_{\text{an}}, C\big)$ is even definable in the field C. (For example, any subset of $\mathbb{C}$ definable in $\big(\mathbb{C}[[t]]_{\text{an}}, \mathbb{C}\big)$ is finite or its complement in $\mathbb{C}$ is finite.) In fact, our induced structure result, Corollary 12.17, is stronger and more general in several ways, for example in also allowing $t^{\mathbb{N}}$ as a distinguished subset of $C[[t]]$. For various reasons it is more convenient to take the fraction field $C((t))$ of $C[[t]]$ as the ambient ring, equipped with its natural valuation to recover $C[[t]]$. For $C = \mathbb{C}$ we obtain [12, Proposition 2] as a special case, as explained in Section 12.2.

Notational and terminological conventions. Throughout d, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$; *ring* means *commutative ring with* 1. From Section 10.3 onwards we consider valued fields. Let K be a *valued* field; it is specified by a valuation ring R of the field K. Let $v : K^\times \to \Gamma$ be a valuation on K with $R = \{a \in K : va \geqslant 0\}$. Here $\Gamma = v(K^\times)$ is the (ordered) value group, and we extend v to a function

$v : K \to \Gamma_\infty = \Gamma \cup \{\infty\}$ by setting $v(0) := \infty$ and we extend the total ordering of Γ to a total ordering on Γ_∞ by $\Gamma < \infty$. It will be convenient to let $\preccurlyeq$, $\asymp$, $\prec$, $\succcurlyeq$, $\succ$, and $\sim$ denote the binary relations on K given for $x, y \in K$ by

$$x \preccurlyeq y \ :\Leftrightarrow\ vx \geqslant vy \ \Leftrightarrow\ x = yz \text{ for some } z \in R,$$
$$x \asymp y \ :\Leftrightarrow\ x \preccurlyeq y \text{ and } y \preccurlyeq x, \qquad x \prec y \ :\Leftrightarrow\ x \preccurlyeq y \text{ and } x \not\asymp y,$$
$$x \succcurlyeq y \ :\Leftrightarrow\ y \preccurlyeq x, \quad x \succ y \ :\Leftrightarrow\ y \prec x, \quad x \sim y \ \Leftrightarrow\ x - y \prec x.$$

We let $\smallo(R)$ be the maximal ideal of R, and let $\operatorname{res} K := R/\smallo(R)$ be the residue field. For $a \in R$ we let $\operatorname{res} a$ be the residue class of a in $\operatorname{res} K$. If we need to indicate dependence on K we write R_K, v_K, Γ_K instead of R, v, Γ. The reason we use the letter R here instead of the more common $\mathcal{O}$ is that in Sections 11.2 and 12.1 we follow [37] in denoting the algebra of affinoid functions on a connected affinoid F by $\mathcal{O}(F)$; see Chapter 11 for context and definitions of these notions.

Model theoretic arguments become important in Chapter 12, although in earlier chapters we already construe various mathematical structures as L-structures for various first-order languages L. We deal only with one-sorted structures, and "$\mathcal{M} \subseteq \mathcal{N}$" indicates that $\mathcal{M}$ is a substructure of $\mathcal{N}$, for L-structures $\mathcal{M}$ and $\mathcal{N}$. (One exception: at the end of Section 12.2 we refer to a 3-sorted structure from [12].)

We cite many results of classical AKE-theory from the exposition [29]. We do so for convenience and do not suggest that the cited facts originate with [29].

9.2 Henselianity

There are a few places where we need "henselianity" outside the usual pattern of a henselian local ring. That is why we give in this section proofs of a few basic facts about henselian pairs, which generalize henselian local rings.

Given a ring R we let $R^\times$ denote the multiplicative group of units of R. The *Jacobson radical* of a ring R is the intersection of the maximal ideals of R. For the Jacobson radical J of R, if $a \in R$ and $a + J \in (R/J)^\times$, then $a \in R^\times$. In this section X and Y are distinct indeterminates and I is an ideal of the ring R.

Lemma 9.1. *Let I be contained in the Jacobson radical of R and let $P(X) \in R[X]$ and $a \in R$ be such that $P'(a) \in R^\times$. Then $P(b) = 0$ for at most one $b \in a + I$.*

Proof. Let $b \in a + I$ and $P(b) = 0$. Then for $\varepsilon \in I$ we have $r \in R$ such that

$$P(b + \varepsilon) \ = \ P(b) + P'(b)\varepsilon + r\varepsilon^2 \ = \ P'(b)\varepsilon + r\varepsilon^2 \ = \ P'(b)\varepsilon\big(1 + rP'(b)^{-1}\varepsilon\big) \ = \ 0,$$

and $P'(b), 1 + rP'(b)^{-1}\varepsilon \in R^\times$, so $\varepsilon = 0$. □

The pair (R, I) is henselian means:

- I is contained in the Jacobson radical of R, equivalently, $1 + I \subseteq R^\times$;
- for all polynomials $P(X) \in R[X]$ and $a \in R$ with $P(a) \in I$ and $P'(a) \in R^\times$ there exists $b \in R$ such that $P(b) = 0$ and $a - b \in I$.

Thus given a maximal ideal $\mathfrak{m}$ of the ring R, the pair $(R, \mathfrak{m})$ is henselian iff R is a henselian local ring in the usual sense.

Lemma 9.2. *Assume $1 + I \subseteq R^\times$. Then the following conditions are equivalent:*

(i) (R, I) *is henselian;*

(ii) each polynomial $1 + X + ea_2X^2 + \cdots + ea_nX^n$ with $n \geqslant 2$, $e \in I$, and $a_2, \ldots, a_n \in R$ has a zero in R (obviously, such a zero lies in $-1 + I$);

(iii) each polynomial $Y^n + Y^{n-1} + ea_2Y^{n-2} + \cdots + ea_n$ with $n \geqslant 2$, $e \in I$, and $a_2, \ldots, a_n \in R$ has a zero in $R^\times$;

(iv) given any polynomial $P(X) \in R[X]$ and $a \in R$, $e \in I$ such that $P(a) = eP'(a)^2$ there exists $b \in R$ such that $P(b) = 0$ and $b - a \in eP'(a)R$.

Proof. (i)⇒(ii) is clear. For (ii)⇔ (iii): use that for $x \in R^\times$ and $y := x^{-1}$, x is a zero in (ii) iff y is a zero in (iii). Now assume (ii) and let P, a, e be as in the hypothesis of (iv). Let $x \in R$ and consider the expansion:

$$\begin{aligned} P(a + x) &= P(a) + P'(a)x + \sum_{i \geqslant 2} P_{(i)}(a)x^i \\ &= eP'(a)^2 + P'(a)x + \sum_{i \geqslant 2} P_{(i)}(a)x^i. \end{aligned}$$

Set $x = eP'(a)y$ where $y \in R$. Then

$$P(a + x) = eP'(a)^2 \left(1 + y + \sum_{i \geqslant 2} ea_iy^i\right)$$

where the $a_i \in R$ do not depend on y. From (ii) we obtain $y \in R$ such that

$$1 + y + \sum_{i \geqslant 2} ea_iy^i = 0.$$

This yields an element $b = a + x = a + eP'(a)y$ as required. This shows (ii) ⇒ (iv), and (iv) ⇒ (i) is clear. □

Lemma 9.3. *Suppose every element of I is nilpotent. Then (R, I) is henselian.*

Proof. Consider a polynomial $P(X) = a + X + \sum_{i=2}^n ea_iX^i$ where $n \geqslant 2$ and

$$a, e, a_2, \ldots, a_n \in R, \quad e^m = 0,\ m \geqslant 1.$$

By induction on m we show that $P(X)$ has a zero in R. The case $m = 1$ being trivial, let $m \geqslant 2$. Then

$$P(-a + eY) = a + (-a + eY) + \sum_{i=2}^n ea_i(-a + eY)^i = e\big(Y + \sum_{i=2}^n a_i(-a + eY)^i\big).$$

An easy computation gives $f, b, b_2, \ldots, b_n \in R$ such that

$$Y + \sum_{i=2}^n a_i(-a + eY)^i = b + Y(1 + ef) + \sum_{i=2}^n e^2b_iY^i.$$

Now use that $1 + ef \in R^\times$ and $(e^2)^{m-1} = 0$. $\square$

Lemma 9.4. *Let J be an ideal of R with $I \subseteq J$. Then the following are equivalent:*

(i) (R, I) *and* $(R/I, J/I)$ *are henselian;*

(ii) (R, J) *is henselian.*

Proof. The condition $1 + J \subseteq R^\times$ is easily seen to be equivalent to the conjunction of $1 + I \subseteq R^\times$ and $1 + (J/I) \subseteq (R/I)^\times$. This gives (ii)⇒(i). Now assume (i), and let $P(X) \in R[X]$ and $a \in R$ with $P(a) \in J$, $P'(a) \in R^\times$. Working modulo I this gives $b \in R$ such that $P(b) \in I$ and $a - b \in J$. Hence $P'(b) - P'(a) \in J$, and thus $P'(b) \in R^\times$, giving $c \in R$ with $P(c) = 0$ and $b - c \in I$. Hence $a - c \in J$. $\square$

Corollary 9.5. *Suppose (R, I) is henselian and J is an ideal of R contained in the nilradical $\sqrt{I}$ of I. Then (R, J) is henselian.*

Proof. Every element of $\sqrt{I}/I$ is nilpotent in R/I, so by Lemmas 9.3 and 9.4 the pair $(R, \sqrt{I})$ is henselian, and so is (R, J). $\square$

Recall also that a local ring R is said to be *henselian* if the pair $(R, \mathfrak{m})$ is henselian, where $\mathfrak{m}$ is the maximal ideal of R.

9.3 Complete ultranormed rings and restricted power series

We introduce here the restricted power series that will define operations on the valuation rings considered in later sections, where we develop an AKE-theory for these valuation rings with these extra operations. The coefficients of these restricted power series will be from a fixed coefficient ring A which is complete with respect to an ultranorm. We begin with defining ultranorms.

Ultranormed abelian groups. Let A be an additively written abelian group. An *ultranorm* on A is a function $a \mapsto |a| : A \to \mathbb{R}^{\geqslant}$ such that for all $a, b \in A$,

- $|a| = 0 \Leftrightarrow a = 0$;
- $|-a| = |a|$;
- $|a + b| \leqslant \max(|a|, |b|)$.

Let A be equipped with the ultranorm $|\cdot|$ on A. We make A a metric space with metric $(a, b) \mapsto |a - b|$. Then A is a topological group with respect to the topology on A induced by this metric. The ultranorm $|\cdot| : A \to \mathbb{R}$ and the group operations $- : A \to A$ and $+ : A \times A \to A$ are uniformly continuous.

In the rest of this subsection A is complete with respect to its ultranorm, that is, complete with respect to the metric above. We now discuss convergence of series with terms in A. Let $(a_i) = (a_i)_{i \in I}$ be a family in A (that is, all $a_i \in A$). We say (a_i) is *summable* if for every ε we have $|a_i| < \varepsilon$ for all but finitely many $i \in I$. In that case the set of $i \in I$ with $a_i \neq 0$ is countable, and there is a unique $a \in A$ such that for every $\varepsilon \in \mathbb{R}^>$ there is a finite $I(\varepsilon) \subseteq I$ with $|a - \sum_{i \in J} a_i| < \varepsilon$ for all finite $J \subseteq I$ with $I(\varepsilon) \subseteq J$; this a is then denoted by $\sum_{i \in I} a_i$ (or $\sum_i a_i$ if I is understood from the context). Instead of saying that (a_i) is summable we also say that $\sum_i a_i$ exists, or that $\sum_i a_i$ converges. Of course, if I is finite, then $\sum_i a_i$ exists and is the usual sum. Here are simple rules, used throughout, for dealing with such (possibly infinite) sums, where $(a_i)_{i \in I}$ is a summable family in A:

- if $c \in \mathbb{R}^{>}$ and $|a_i| \leqslant c$ for all i, then $|\sum_i a_i| \leqslant c$;
- $(-a_i)$ is summable with $\sum_i -a_i = -\sum_i a_i$;
- if $(b_i)_{i \in I}$ is also a summable family in A, then so is $(a_i + b_i)$ with

$$\sum_i a_i + b_i \;=\; \sum_i a_i + \sum_i b_i;$$

- if $i \mapsto \lambda(i) : I \to \Lambda$ is a bijection and $(b_\lambda)_{\lambda \in \Lambda}$ is a family in A with $a_i = b_{\lambda(i)}$ for all $i \in I$, then $\sum_\lambda b_\lambda$ exists and equals $\sum_i a_i$;
- if the family $(a_j)_{j \in J}$ in A is also summable with $I \cap J = \emptyset$, then $(a_k)_{k \in I \cup J}$ is summable with $\sum_k a_k = \sum_i a_i + \sum_j a_j$;
- if $I = \dot{\bigcup}_{\lambda \in \Lambda} I_\lambda$ (disjoint union), then $\sum_{i \in I_\lambda} a_i$ exists for all $\lambda \in \Lambda$, and $\sum_\lambda \big(\sum_{i \in I_\lambda} a_i\big)$ exists and equals $\sum_{i \in I} a_i$.

Suppose E is a closed subgroup of A. Then

$$|a + E| \;:=\; \inf_{e \in E} |a + e| \qquad (a \in A)$$

yields an ultranorm on the quotient group A/E with respect to which A/E is complete; we call it the *quotient norm of* A/E. If the family (a_i) in A is summable, then so is the family $(a_i + E)$ in A/E with its quotient norm, and

$$\Big(\sum_i a_i\Big) + E \;=\; \sum_i (a_i + E).$$

Ultranormed rings. Let A be a ring. An *ultranorm on* A is a function

$$a \mapsto |a| \;:\; A \to \mathbb{R}^{\geqslant}$$

such that for all $a, b \in A$,

- $|a| = 0 \Leftrightarrow a = 0$, $|1| = |-1| = 1$;
- $|a + b| \leqslant \max(|a|, |b|)$;
- $|ab| \leqslant |a| \cdot |b|$.

Let A be equipped with the ultranorm $|\cdot|$ on A. Then $|-a| = |a|$ for all $a \in A$, so $|\cdot|$ is an ultranorm on the underlying additive group of A. The function $\cdot : A \times A \to A$ is continuous. If A is complete with respect to its ultranorm and $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are summable families in A, then $(a_i b_j)_{(i,j) \in I \times J}$ is summable, with $(\sum_i a_i)(\sum_j b_j) = \sum_{(i,j)} a_i b_j$.

From now on in this part of the thesis, A is a ring with $1 \neq 0$, equipped with an ultranorm $|\cdot|$ such that $|a| \leqslant 1$ for all $a \in A$, and A is complete with respect to its ultranorm.

If $a \in A$ and $|a| < 1$, then $\sum_n a^n$ exists, with

$$(1 - a) \sum_n a^n \;=\; 1.$$

We have the ideal $\smallo(A) := \{a \in A : |a| < 1\}$, and set $\overline{A} := A/\smallo(A)$, with the canonical ring morphism $a \mapsto \overline{a} = a + \smallo(A) : A \to \overline{A}$. We saw that $1 + \smallo(A)$ consists entirely of units of A. Thus $a \in A$ is a unit of A iff $\overline{a}$ is a unit of $\overline{A}$. In particular, $\smallo(A)$ is contained in the Jacobson radical of A. The completeness assumption now yields Hensel's Lemma as stated in [29, Section 2.2]: the pair $(A, \smallo(A))$ is henselian. It follows that $(A, \sqrt{\smallo(A)})$ is also henselian.

Passing to A/I. Suppose the proper ideal I of A is closed. Then the quotient norm of the quotient group A/I is an ultranorm on the ring A/I. Equipping A/I with the quotient norm, the canonical map $A \to A/I$ is norm decreasing, and $\smallo(A/I)$ is the image of $\smallo(A)$ under this canonical map. Next we describe some ways of extending A to a larger complete ultranormed ring.

From A to $A[[\xi]]$. Let $\xi = (\xi_1, \dots, \xi_m)$ be a tuple of m distinct indeterminates. Here and below, when using an expression like $f = \sum_\mu a_\mu \xi^\mu$ for a series in $A[[\xi]]$, we assume the coefficients a_μ are all in A, and $\mu = (\mu_1, \dots, \mu_m)$ ranges over the elements of $\mathbb{N}^m$. We also set $\xi^\mu := \xi_1^{\mu_1} \cdots \xi_n^{\mu_m}$ and $|\mu| := \mu_1 + \cdots + \mu_m$. The latter conflicts with our notation for the ultranorm on A, but in practice no confusion will arise. For $f = \sum_\mu a_\mu \xi^\mu \in A[[\xi]]$ set $\|f\| := \max_\mu |a_\mu| 2^{-|\mu|}$. This gives an ultranorm $\|\cdot\|$ on the ring $A[[\xi]]$ extending the ultranorm on A.

Lemma 9.6. *$A[[\xi]]$ is complete with respect to the ultranorm $\|\cdot\|$.*

Proof. Let (f_n) be a cauchy sequence with respect to this ultranorm, with

$$f_n \ = \ \sum_\mu a_{n\mu} \xi^\mu.$$

For $\varepsilon > 0$, take $N(\varepsilon) \in \mathbb{N}$ such that $\|f_m - f_n\| < \varepsilon$ for all $m, n > N(\varepsilon)$. Now fix μ. Then for $m, n > N(\varepsilon)$,

$$|a_{m\mu} - a_{n\mu}| \ < \ \varepsilon \cdot 2^{|\mu|},$$

so $(a_{n\mu})$ is a cauchy sequence in A. Setting $a_\mu := \lim_{n\to\infty} a_{n\mu}$ gives $f := \sum_\mu a_\mu \xi^\mu$ in $A[[\xi]]$, with $|a_\mu - a_{n\mu}| \leqslant \varepsilon \cdot 2^{|\mu|}$ for $n > N(\varepsilon)$, so $\|f - f_n\| \leqslant \varepsilon$ for $n > N(\varepsilon)$. Hence $f_n \to f$ in $A[[\xi]]$ as $n \to \infty$. $\square$

We have an internal direct sum $\smallo(A[[\xi]]) = \smallo(A) \oplus (\xi_1, \dots, \xi_m)A[[\xi]]$ of A-modules. For any real number $r > 1$, the above material in this subsection goes through when replacing 2 by r, and yields a complete ultranorm with the same neighborhoods of $0 \in A[[\xi]]$, so the same (ring) topology on $A[[\xi]]$ as for $r = 2$.

Restricted power series over an ultranormed ring. For distinct indeterminates $Y_1, \dots, Y_n$ we let $A\langle Y\rangle = A\langle Y_1, \dots, Y_n\rangle$ be the subalgebra of the A-algebra $A[[Y_1, \dots, Y_n]]$ consisting of the series $\sum_\nu a_\nu Y^\nu$ with $a_\nu \to 0$ as $|\nu| \to \infty$. Here and below, when using an expression like $\sum_\nu a_\nu Y^\nu$ for a series in $A\langle Y\rangle$ it is assumed that $a_\nu \to 0$ as $|\nu| \to \infty$. We extend $|\cdot|$ on A to an ultranorm on the ring $A\langle Y\rangle$ by

$$\Big|\sum_\nu a_\nu Y^\nu\Big| \ := \ \max_\nu |a_\nu|,$$

so with respect to this ultranorm, $A\langle Y\rangle$ is complete and $A[Y]$ is dense in it. Note that for $a_1, \dots, a_n \in \smallo(A)$ we have $|a_1 Y_1 + \cdots + a_n Y_n| < 1$, so $1 + a_1 Y_1 + \cdots + a_n Y_n$ is a unit of the ring $A\langle Y\rangle$.

For $f = \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$, the family $(a_\nu Y^\nu)$ in $A\langle Y\rangle$ is in fact summable with sum f. If $|ab| = |a|\cdot|b|$ for all $a, b \in A$, then $|fg| = |f|\cdot|g|$ for all $f, g \in A\langle Y\rangle$. For any $y = (y_1, \dots, y_n) \in A^n$ we have the evaluation map

$$f = \sum_\nu a_\nu Y^\nu \mapsto f(y) := \sum_\nu a_\nu y^\nu \ :\ A\langle Y\rangle \to A,$$

which is a A-algebra morphism with $|f(y)| \leqslant |f|$ for all $y \in A^n$. If $(f_i)_{i\in I}$ is a summable family in $A\langle Y\rangle$ and $y \in A^n$, then $\sum_i f_i(y)$ exists in A and equals $(\sum_i f_i)(y)$. The obvious inclusion of $A[[Y_1, \dots, Y_m]]$ in $A[[Y_1, \dots, Y_n]]$ for $m \leqslant n$ restricts to an inclusion of $A\langle Y_1, \dots, Y_m\rangle$ in $A\langle Y_1, \dots, Y_n\rangle$. For $f = f(Y) \in A\langle Y\rangle$ we have unique $f_j \in A\langle Y_1, \dots, Y_j\rangle$ for $j = 0, \dots, n$ such that

$$f(Y) \ = \ f_0 + Y_1 f_1 + \cdots + Y_n f_n.$$

Substitution. Besides $Y = (Y_1, \dots, Y_n)$, let $X = (X_1, \dots, X_m)$ also be a tuple of distinct indeterminates. Let $f = \sum_\mu a_\mu X^\mu \in A\langle X\rangle$ with $\mu = (\mu_1, \dots, \mu_m)$ ranging over $\mathbb{N}^m$, and $g_1, \dots, g_m \in A\langle Y\rangle$. Then $|a_\mu g_1^{\mu_1} \cdots g_m^{\mu_m}| \leqslant |a_\mu| \to 0$ as $|\mu| \to \infty$, so

$$f(g_1, \dots, g_m) \ := \ \sum_\mu a_\mu g_1^{\mu_1} \cdots g_m^{\mu_m} \in A\langle Y\rangle,$$

and for fixed $g = (g_1, \dots, g_m) \in A\langle Y\rangle^m$ the map $f \mapsto f(g) : A\langle X\rangle \to A\langle Y\rangle$ is an A-algebra morphism with $|f(g)| \leqslant |f|$ and $f(g)(y) = f(g(y))$ for $y \in A^n$. Moreover, if (f_i) is a summable family in $A\langle X\rangle$ and $g \in A\langle Y\rangle^m$, then $\sum_i f_i(g)$ exists in $A\langle Y\rangle$ and equals $(\sum_i f_i)(g)$. It follows that the above kind of composition is associative in the following sense: let $Z = (Z_1, \dots, Z_p)$ be a third tuple of distinct indeterminates, $p \in \mathbb{N}$, and $h = (h_1, \dots, h_n) \in A\langle Z\rangle^n$. Then

$$\big(f(g)\big)(h) \ = \ f\big(g_1(h), \dots, g_m(h)\big) \text{ in } A\langle Z\rangle.$$

From now on $X_1, X_2, X_3, \dots, Y_1, Y_2, Y_3, \dots$ (two infinite sequences) are distinct indeterminates, and unless specified otherwise,

$$X := (X_1, \dots, X_m), \qquad Y := (Y_1, \dots, Y_n).$$

The natural $A[[X]]$-algebra isomorphism $A[[X]][[Y]] \to A[[X, Y]]$ restricts to the norm preserving $A\langle X\rangle$-algebra isomorphism $A\langle X\rangle\langle Y\rangle \to A\langle X, Y\rangle$ given by

$$\sum_\nu f_\nu Y^\nu \mapsto \sum_\nu f_\nu Y^\nu$$

where $f_\nu \in A\langle X\rangle$ for all ν and $f_\nu \to 0$ as $|\nu| \to \infty$, with righthand and lefthand side interpreted naturally in $A\langle X\rangle\langle Y\rangle$ and $A\langle X, Y\rangle$ respectively. We identify $A\langle X\rangle\langle Y\rangle$ and $A\langle X, Y\rangle$ via this isomorphism.

Adjoining a zero of a monic polynomial to A. Let $p = p(T) \in A[T]$ be a monic polynomial of degree $d \geqslant 1$ over A, so $|p| = 1$ as an element of $A\langle T\rangle$.

Lemma 9.7. *For all $f \in A\langle T\rangle$ we have $|pf| = |f|$. Moreover, $pA\langle T\rangle$ is a proper ideal of $A\langle T\rangle$ and is closed in $A\langle T\rangle$.*

Proof. For $f = \sum_n a_n T^n \in A[T]^{\neq}$, take n maximal with $|a_n| = |f|$, and note that then the coefficient of T^{d+n} in pf is $a_n + b$ with $|b| < |a_n|$, so $|a_n + b| = |a_n| = |f|$. The rest follows easily. $\square$

Lemma 9.8. *Let $f \in A\langle T\rangle$. Then there are unique $q \in A\langle T\rangle$ and $r \in A[T]$ with $\deg r < d$ such that $f = qp + r$; moreover, $|f| = \max(|q|, |r|)$ for these q, r.*

Proof. For each n we have $T^n = q_n p + r_n$ with $q_n, r_n \in A[T]$ and $\deg r_n < d$. Thus for $f = \sum_n a_n T^n \in A\langle T\rangle$ we have $f = qp + r$ with $q = \sum_n a_n q_n \in A\langle T\rangle$ and $r = \sum_n a_n r_n \in A[T]$ with $\deg r < d$, and $|f| = \max(|q|, |r|)$ for these q, r.

Uniqueness holds because for $g \in A\langle T\rangle$ with $gp \in A[T]$, $\deg gp < d$, we have $g = 0$ by the proof of Lemma 9.7. ☐

Corollary 9.9. *The composition $A[T] \to A\langle T\rangle \to A\langle T\rangle/pA\langle T\rangle$, with inclusion on the left and the canonical map on the right, is surjective and has kernel $pA[T]$.*

Proof. Lemma 9.8 gives surjectivity. The uniquness in that lemma and division with remainder in $A[T]$ (by p) yields kernel $pA[T]$. ☐

The composition map of the lemma induces a ring isomorphism

$$A[T]/pA[T] \to A\langle T\rangle/pA\langle T\rangle$$

by means of which we identify these two rings. We also identify A with a subring of $A[T]/pA[T]$ via the injective ring morphism $A \to A[T] \to A[T]/pA[T]$. Hence

$$A[T]/pA[T] \;=\; A[t_p] \;=\; A \oplus At_p \oplus \cdots \oplus At_p^{d-1} \text{(internal direct sum of } A\text{-modules)}$$

where t_p is the image of T in $A[T]/pA[T]$ under the canonical map $A[T]/pA[T]$.

Lemma 9.10. *The quotient norm of $A[t_p] = A\langle T\rangle/pA\langle T\rangle$ satisfies*

$$|a_0 + a_1 t_p + \cdots + a_{d-1} t_p^{d-1}| \;=\; \max_{i<d} |a_i| \quad \text{for } a_0, \ldots, a_{d-1} \in A,$$

and so extends the norm of A, and $|t_p^i| = 1$ for $i < d$.

Proof. Let $a_0, \ldots, a_{d-1} \in A$ and $a := a_0 + a_1 T + \cdots + a_{d-1} T^{d-1} \in A\langle T\rangle$. Then $|a| = \max_{i<d} |a_i|$. Let $f \in A\langle T\rangle$; it is enough to show that then $|a + pf| \geqslant |a|$. Since $|pf| = |f|$, this inequality certainly holds if $|a| \neq |f|$, so assume $|a| = |f|$. Then the proof of Lemma 9.7 yields that for some n the norm of the coefficient of T^{d+n} in pf equals $|a|$, and so $|a + pf| \geqslant |a|$. ☐

We shall denote $A[t_p]$ equipped with this quotient norm by $A\langle t_p\rangle$.

Another norm on $A[t_p]$.

Let p be as in the previous subsection, but now assume also that $p = T^d + \sum_{i<d} p_i T^i$ with $p_i \in \smallo(A)$ for all $i < d$. It would then be reasonable to have an ultranorm on $A[t_p]$ for which t_p has norm < 1, differing therefore from the quotient norm of Lemma 9.10 if $d \geqslant 2$. To accomplish this we use $A[[T]]$ instead of $A\langle T\rangle$, with the complete ultranorm on $A[[T]]$ given by $\|f\| := \max_n |a_n| 2^{-n}$ for $f = \sum_n a_n T^n \in A[[T]]$. Thus $2^{-d} \leqslant \|p\| < 1$, and adapting the argument in the proof of Lemma 9.7 now gives:

Lemma 9.11. *For all $f \in A[[T]]^{\neq}$ we have $2^{-d}\|f\| \leqslant \|pf\| < \|f\|$. Moreover, $pA[[T]]$ is a proper ideal of $A[[T]]$ and is closed in $A[[T]]$.*

Next an analogue of Lemma 9.8:

Lemma 9.12. *Let $f \in A[[T]]$. Then there are unique $q \in A[[T]]$ and $r \in A[T]$ with $\deg r < d$ such that $f = qp + r$.*

Proof. Let I be the ideal $(p_0, \dots, p_{d-1})$ of A. We claim that for each n,

$$T^n \;=\; q_n p + r_n \;\text{ with } q_n, r_n \in A[T],\; r_n \in \sum_{i<d} I^{[n/d]} T^i.$$

Let $md \leqslant n < (m+1)d$, so $[n/d] = m$. Then from $T^{md} = (p - \sum_{i<d} p_i T^i)^m$ we obtain $T^n = q_{n1}p + \sum_{j<n} r_{jn} T^j$ with $q_{n1} \in A[T]$ and all $r_{jn} \in I^m$. Using for $j < n$ that $T^j = q_j^* p + r_j^*$ with $q_j^*, r_j^* \in A[T]$, $\deg r_j^* < d$ this gives

$$T^n \;=\; \big(q_{n1} + \sum_{j<n} r_{jn} q_j^*\big)p + \sum_{j<n} r_{jn} r_j^*,$$

so the claim holds with $q_n = q_{n1} + \sum_{j<n} r_{jn} q_j^*$ and $r_n = \sum_{j<n} r_{jn} r_j^*$.

Set $\rho := \max_{i<d} |p_i|$. Then for q_n, r_n as in the claim we have $\|r_n\| \leqslant \rho^{[n/d]}$, which in view of Lemma 9.11 gives $\|q_n\| \leqslant 2^d \max(2^{-n}, \rho^{[n/d]}) \to 0$ as $n \to \infty$. Thus for $f = \sum_n a_n T^n \in A[[T]]$ we have $f = qp + r$ with $q = \sum_n a_n q_n \in A[[T]]$ and $r = \sum_n a_n r_n \in A[T]$ with $\deg r < d$. Uniqueness holds because for $g \in A[[T]]$ with $gp \in A[T]$, $\deg gp < d$, we have $g = 0$ by the argument in the proof of Lemma 9.7. □

Corollary 9.13. *The composition $A[T] \to A[[T]] \to A[[T]]/pA[[T]]$, with inclusion on the left and the canonical map on the right, is surjective and has kernel $pA[T]$.*

Proof. Lemma 9.12 gives surjectivity. The uniqueness in that lemma and division with remainder in $A[T]$ (by p) yields kernel $pA[T]$. □

The composition map of the lemma induces a ring isomorphism

$$A[T]/pA[T] \to A[[T]]/pA[[T]]$$

by means of which we identify these two rings. We also identify A with a subring of $A[T]/pA[T]$ via the injective ring morphism $A \to A[T] \to A[T]/pA[T]$. Hence

$$A[[T]]/pA[[T]] \;=\; A[t_p] \;=\; A\langle T\rangle/pA\langle T\rangle.$$

as rings. We denote the ring $A[[T]]/pA[[T]]$ equipped with the quotient norm $\|\cdot\|$ coming from $(A[[T]], \|\cdot\|)$ by $A[[t_p]]$. Thus $\|t_p\| \leqslant 1/2$. Moreover:

Lemma 9.14. *Let $a, a_0, \dots, a_{d-1} \in A$. Then*

(i) $|a| < 1 \Leftrightarrow \|a\| < 1$;

(ii) $\|a_0 + a_1 t_p + \cdots + a_{d-1} t_p^{d-1}\| \;\geqslant\; 2^{-2d} \max_{i<d} |a_i|$;

(iii) $\mathcal{O}(A[[t_p]]) \;=\; \mathcal{O}(A) + t_p A + \cdots + t_p^{d-1} A$.

Proof. For (i), first note that $\|a\| \leqslant |a|$. Next, observe that for $f \in A[[T]]$ the constant term of $a + pf$ is $a + \varepsilon$ with $|\varepsilon| < 1$, so if $|a| = 1$, then $|a + \varepsilon| = 1$ and thus $\|a\| = 1$ in $A[[t_p]]$.

For (ii), set $b := a_0 + a_1 T + \cdots + a_{d-1} T^{p-1} \in A[[T]]$, so $\|b\| \geqslant 2^{-d} \max_{i<d} |a_i|$. Let $f \in A[[T]]$; it is enough to show that then $\|b + pf\| \geqslant 2^{-d}\|b\|$. If $\|b\| \neq \|pf\|$, then $\|b + pf\| \geqslant \|b\|$, so the inequality holds. Assume $\|b\| = \|pf\|$. Then $\|b\| \leqslant \|f\|$, and the proof of Lemma 9.11 gives $\|b+pf\| \geqslant 2^{-d}\|f\|$, hence $\|b+pf\| \geqslant 2^{-d}\|b\|$.

As to (iii), this is clear from $\|t_p\| < 1$ and (i). $\square$

Therefore the norms $|\cdot|$ and $\|\cdot\|$ on the ring $A[t_p]$ are equivalent:

$$2^{-2d}|a| \ \leqslant\ \|a\| \ \leqslant\ |a| \qquad (a \in A[t_p]),$$

and so induce the same ring topology on $A[t_p]$. It follows that $A\langle t_p\rangle\langle Y\rangle$ and $A[[t_p]]\langle Y\rangle$ have the same underlying ring, denoted by $A[t_p]\langle Y\rangle$, and equivalent norms: $2^{-2d}|f| \leqslant \|f\| \leqslant |f|$ for $f \in A[t_p]\langle Y\rangle$, where $|\cdot|$, respectively $\|\cdot\|$, denotes the norm of $A\langle t_p\rangle\langle Y\rangle$, respectively of $A[[t_p]]\langle Y\rangle$. Moreover,

$$A[t_p]\langle Y\rangle \ = \ A\langle Y\rangle \oplus A\langle Y\rangle t_p \oplus \cdots \oplus A\langle Y\rangle t_p^{d-1},$$

an internal direct sum of $A\langle Y\rangle$-modules, and for $f_0, \dots, f_{d-1} \in A\langle Y\rangle$ we have

$$\begin{aligned} |f_0 + f_1 t_p + \cdots + f_{d-1} t_p^{d-1}| \ &= \ \max_{i<d} |f_i|, \\ \|f_0 + f_1 t_p + \cdots + f_{d-1} t_p^{d-1}\| \ &\geqslant \ 2^{-2d} \max_{i<d} |f_i|. \end{aligned}$$

Division with Remainder. Let $n \geqslant 1$, set $Y' := (Y_1, \dots, Y_{n-1})$. The inclusion $A\langle Y'\rangle[Y_n] \subseteq A\langle Y'\rangle\langle Y_n\rangle = A\langle Y\rangle$ makes $A\langle Y'\rangle[Y_n]$ a subring of $A\langle Y\rangle$.

Lemma 9.15. *Let $f \in A\langle Y'\rangle[Y_n]$ be monic of degree d and $g \in A\langle Y\rangle$. Then there are unique $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with $\deg_{Y_n} r < d$ such that $g = qf + r$. Moreover, $|g| = \max(|q|, |r|)$ for these q, r.*

Proof. This is Lemma 9.8 applied to $A\langle Y'\rangle$ in the role of A. $\square$

Consider $A\langle X, Y_1, \dots, Y_{j-1}\rangle[Y_j]$ likewise as a subring of $A\langle X, Y\rangle$ for $j = 1, \dots, n$. By a straightforward induction on n the previous lemma gives:

Lemma 9.16. *Let $f_j \in A\langle X, Y_1, \dots, Y_{j-1}\rangle[Y_j]$ be monic of degree d_j in Y_j for $j = 1, \dots, n$. Then*

$$A\langle X, Y\rangle \ = \ (f_1, \dots, f_n)A\langle X, Y\rangle + \bigoplus_{(j_1,\dots,j_n)} A\langle X\rangle Y_1^{j_1} \cdots Y_n^{j_n}$$

where $(j_1, \dots, j_n)$ ranges over the elements of $\mathbb{N}^n$ with $j_1 < d_1, \dots, j_n < d_n$.

Corollary 9.17. *Let $m = n$ and $f(X) \in A\langle X\rangle$. Then*

$$f(X) - f(Y) \in (X_1 - Y_1, \dots, X_n - Y_n)A\langle X, Y\rangle.$$

Proof. By Lemma 9.16 we have $f(X) - f(Y) = \sum_{j=1}^n (X_j - Y_j)q_j + r$ with all q_j in $A\langle X, Y\rangle$ and $r \in A\langle X\rangle$. Substituting X_j for Y_j gives $0 = r$. $\square$

We extend $a \mapsto \overline{a} : A \to \overline{A}$ to the ring morphism

$$f = \sum_\nu a_\nu Y^\nu \ \mapsto \overline{f} := \sum_\nu \overline{a_\nu} Y^\nu \ : \ A\langle Y\rangle \to \overline{A}[Y],$$

whose kernel is $o(A\langle Y\rangle) := \{f \in A\langle Y\rangle : \ |f| < 1\}$. Moreover,

$$\overline{f(g_1,\dots,g_m)} = \overline{f}(\overline{g_1},\dots,\overline{g_m}), \qquad (f \in A\langle X\rangle,\ g_1,\dots,g_m \in A\langle Y\rangle).$$

For $d \in \mathbb{N}$, call $f \in A\langle Y\rangle$ *regular in* Y_n *of degree* d if $\overline{f} = f_0 + f_1 Y_n + \cdots + f_d Y_n^d$ with $f_0,\dots,f_d \in \overline{A}[Y']$ and f_d a unit in $\overline{A}[Y']$. We now extend Lemma 9.15:

Proposition 9.18 (Weierstrass Division)**.** *Suppose $f \in A\langle Y\rangle$ is regular in Y_n of degree d and $g \in A\langle Y\rangle$. Then there are $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with*

$$g \ = \ qf + r, \qquad \deg_{Y_n} r \ < \ d, \qquad |g| = \max(|q|,|r|).$$

Proof. Multiplying f by a unit of $A\langle Y'\rangle$ we arrange that $\overline{f} \in \overline{A}[Y]$ is monic in Y_n of degree d. Hence $f = f_0 + E$ where $f_0 \in A\langle Y'\rangle[Y_n]$ is monic of degree d in Y_n and $E \in A\langle Y\rangle,\ |E| < 1$. Now $g = q_0 f_0 + r_0$ with $q_0 \in A\langle Y\rangle$ and $r_0 \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r_0 < d$ and $|g| = \max(|q_0|,|r_0|)$, so $g = q_0 f + r_0 + g_1$ with $g_1 = -Eq_0$, and thus $|g_1| \leqslant |E||g|$. With g_1 in the role of g and iterating:

$$\begin{aligned}
g \ &= \ q_0 f + r_0 + g_1, & g_1 \ &= \ -Eq_0, & |g_1| &\leqslant |E||g|,\\
g_1 \ &= \ q_1 f + r_1 + g_2, & g_2 \ &= \ -Eq_1, & |g_2| &\leqslant |E|^2|g|,\\
\dots \ &= \ \dots\\
\dots \ &= \ \dots\\
g_k \ &= \ q_k f + r_k + g_{k+1}, & g_{k+1} \ &= \ -Eq_k, & |g_{k+1}| &\leqslant |E|^{k+1}|g|,\\
\dots \ &= \ \dots
\end{aligned}$$

where $q_k \in A\langle Y\rangle,\ r_k \in A\langle Y'\rangle[Y_n],\ \deg_{Y_n} r_k < d$ and $|g_k| = \max(|q_k|,|r_k|)$. It follows that $g_k, q_k, r_k \to 0$ as $k \to \infty$. Thus we can add the right and left-hand sides in the equalities above to obtain $g = qf + r$ where $q := \sum_k q_k \in A\langle Y\rangle$ and $r := \sum_k r_k \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r < d$, so $|g| = \max(|q|,|r|)$. ☐

Corollary 9.19 (Weierstrass Preparation)**.** *Suppose $f \in A\langle Y\rangle$ is regular in Y_n of degree d. Then for some unit u of $A\langle Y\rangle$ we have: $uf \in A\langle Y'\rangle[Y_n]$, and uf is monic of degree d in Y_n.*

Proof. We have $Y_n^d = qf + r$ with $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r < d$. Hence $Y_n^d - \overline{r} = \overline{q}\,\overline{f}$ in $\overline{A}[Y]$, so $\overline{q}$ is a unit of $\overline{A}[Y']$, hence q is a unit of $A\langle Y\rangle$, and thus $u := q$ has the desired property. ☐

A somewhat twisted argument also gives uniqueness in the last two results:

Corollary 9.20. *Let $f \in A\langle Y\rangle$ be regular in Y_n of degree d. Then there is only one pair (q,r) with $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with $g \ = \ qf + r$ and $\deg_{Y_n} r < d$. There is also only one unit u of $A\langle Y\rangle$ such that $uf \in A\langle Y'\rangle[Y_n]$, and uf is monic of degree d in Y_n.*

Proof. By Corollary 9.19 (just the existence of u), the uniqueness of (q,r) follows from the uniqueness in Lemma 9.15. Next, the uniqueness of u follows from the proof of Corollary 9.19 and the uniqueness in Proposition 9.18. ☐

Besides $n \geqslant 1$ we now also assume $d \geqslant 1$. Under an extra assumption on $\overline{A}$ (see Lemma 9.21) we can apply automorphisms to arrange regularity in Y_n. Set

$$T_d(Y) \ := \ \big(Y_1 + Y_n^{d^{n-1}},\dots,Y_{n-1} + Y_n^d, Y_n\big),$$

which gives a norm preserving automorphism $f(Y) \mapsto f\big(T_d(Y)\big)$ of the A-algebra $A\langle Y\rangle$ with inverse $g(Y) \mapsto g\big(T_d^{-1}(Y)\big)$, where

$$T_d^{-1}(Y) \ := \ \big(Y_1 - Y_n^{d^{n-1}}, \dots, Y_{n-1} - Y_n^d, Y_n\big).$$

Lemma 9.21. *Assume $\overline{A}$ is a field. Let $f \in A\langle Y\rangle$ be such that $\overline{f} \neq 0$ in $\overline{A}[Y]$, and $d > \deg \overline{f}$. Then $f\big(T_d(Y)\big)$ is regular in Y_n of some degree.*

Proof. With $f = \sum_\nu a_\nu Y^\nu$, let $(\mu_1, \dots, \mu_n)$ be lexicographically largest among the $\nu \in \mathbb{N}^n$ for which $\overline{a_\nu} \neq 0$. A straightforward computation shows that then for $\ell := \mu_1 d^{n-1} + \cdots + \mu_{n-1} d + \mu_n$ we have

$$\overline{f}\big(T_d(Y)\big) \ = \ \overline{a_\mu} Y_n^\ell + \text{terms in } \overline{A}[Y] \text{ of degree} < \ell \text{ in } Y_n.$$

Thus $f\big(T_d(Y)\big)$ is regular in Y_n of degree ℓ. $\square$

CHAPTER 10

Rings with A-analytic structure

10.1 A-rings

Given a ring R and a set E we have the ring R^E of R-valued functions on E, where the ring operations are given pointwise. A *ring with A-analytic structure* is a ring R together with a ring morphism

$$\iota_n \ : \ A\langle Y_1, \dots, Y_n\rangle \to \text{ ring of } R\text{-valued functions on } R^n$$

for every n, with the following properties:

(A1) $\iota_n(Y_k)(y_1, \dots, y_n) = y_k$, for $k = 1, \dots, n$ and $y = (y_1, \dots, y_n) \in R^n$;

(A2) ι_{n+1} extends ι_n where we identify in the obvious way a function $R^n \to R$ with a function $R^{n+1} \to R$ that does not depend on the $(n+1)$th coordinate;

(A3) for $f, g_1, \dots, g_n \in A\langle Y\rangle$ and $y \in R^n$ we have

$$f(g_1, \dots, g_n)(y) \ = \ f\big(g_1(y), \dots, g_n(y)\big).$$

In the last clause $h(y) := \iota_n(h)(y)$ for $h \in A\langle Y\rangle$ and $y \in R^n$, a notational convention that will be in force from now on. In other words, each $h \in A\langle Y\rangle$ defines a function $R^n \to R$ that we also denote by h. For $n = 0$ the above gives the ring morphism $\iota_0 : A \to R$ upon identifying a function $R^0 \to R$ with its only value, and so R is an A-algebra with structural morphism ι_0. Accordingly we denote for $a \in A$ the element $\iota_0(a)$ of R also by a when no confusion is likely. Simple example of a ring with A-analytic structure: A itself with $\iota_n(f)(y) := f(y)$ for $f \in A\langle Y\rangle$ and $y \in A^n$.

We abbreviate the expression *ring with A-analytic structure* to *A-analytic ring*, or just *A-ring*. A good feature of the above is that the A-rings naturally form an *equational class* (which is not the case for the narrower notion of rings with analytic A-structure defined in [27], although there the third clause has a weaker form than here.) To back this up, we introduce the language $\mathcal{L}^A$ of A-rings: it is the language $\{0, 1, -, +, \cdot\}$ of rings augmented by an n-ary function symbol for each $f \in A\langle Y\rangle = A\langle Y_1, \dots, Y_n\rangle$, to be denoted also by f. We construe any A-ring R in the obvious way as an $\mathcal{L}^A$-structure, with f as above naming the function $y \mapsto f(y) : R^n \to R$, so the A-rings are exactly the models of an equational $\mathcal{L}^A$-theory, and for any $\mathcal{L}^A$-term $t(Z_1, \dots, Z_n)$ there is an $f \in A\langle Y\rangle$ such that $t(z) = f(z)$ for every A-ring R and $z \in R^n$.

Example. Let A_0 be a ring with $1 \neq 0$ and $A := A_0[[t]]$, the power series ring in one variable t over A_0, with the (complete) ultranorm given by $|f| = 2^{-n}$ for $f \in t^nA \setminus t^{n+1}A$. Let $\iota : A_0 \to \boldsymbol{k}$ be a ring morphism into a field $\boldsymbol{k}$, let Γ be an ordered abelian group with a distinguished element $1 > 0$. We identify $\mathbb{Z}$ with its image in Γ via $k \mapsto k \cdot 1$, which makes $\mathbb{Z}$ an ordered subgroup of Γ. (We do not assume here that 1 is the least positive element of Γ.) This yields the Hahn field $K = \boldsymbol{k}((t^{\Gamma}))$ with its valuation ring $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]] \supseteq \boldsymbol{k}[[t]]$. Now ι extends to the ring morphism $A \to \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$,

$$\sum_n c_n t^n \mapsto \sum_n \iota(c_n) t^n \in \boldsymbol{k}[[t]] \quad \text{(with all } c_n \in A_0\text{)}.$$

We have a natural A-analytic structure (ι_n) on $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$, where ι_0 is the above ring morphism $A \to \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$, and more generally, for $f = \sum a_\nu Y^\nu$ in $A\langle Y\rangle$ and $y \in (\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]])^n$,

$$\iota_n(f)(y) \ := \ \sum_\nu \iota_0(a_\nu) y^\nu \in k[[t^{\Gamma^{\geqslant}}]].$$

(To verify (A3) in this case, note that if (f_i) is a summable family in $A\langle Y\rangle$ and $y \in (\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]])^n$, then $\sum_i f_i(y)$ exists in $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ and equals $(\sum_i f_i)(y)$.) Note that $\iota_0(A)$ is the subring $\iota(A_0)[[t]]$ of $\boldsymbol{k}[[t]]$.

Returning to the general setting, let R be an A-ring. Among its units are clearly the elements $1 + a_1 y_1 + \cdots + a_n y_n$ for $a_1, \dots, a_n \in \mathcal{O}(A)$ and $y_1, \dots, y_n \in R$.

Any ideal I of R yields a *congruence relation* for the A-analytic structure of R. This means: for any $f \in A\langle Y\rangle$ and any $x, y \in R^n$ with $x \equiv y \mod I$ (that is, $x_1 - y_1, \dots, x_n - y_n \in I$), we have $f(x) \equiv f(y) \mod I$, an immediate consequence of Corollary 9.17. Thus R/I is an A-ring, given by

$$f(y_1 + I, \dots, y_n + I) \ := \ f(y_1, \dots, y_n) + I \qquad (f \in A\langle Y\rangle,\ (y_1, \dots, y_n) \in R^n).$$

This construal of R/I as an A-ring is part of our goal of developing some algebra for A-rings analogous to ordinary facts about rings. But we need some extra notational flexibility in dealing with indeterminates, as we already tacitly used in this argument about R/I: we do not want to be tied down to the particular sequence of indeterminates $Y_1, Y_2, \dots$ used in the definition of *A-analytic structure*. Namely, for any tuple $Z = (Z_1, \dots, Z_n)$ of distinct indeterminates, not necessarily among the $X_1, X_2, \dots, Y_1, Y_2, \dots$, any $f = f(Z) = \sum_\nu a_\nu Z^\nu \in A\langle Z\rangle$ and $z \in R^n$ we set $f(z) := \big(\iota_n f(Y)\big)(z)$, where $f(Y) := \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$.

This is in harmony with other notational conventions: Let $V, Z_1, \dots, Z_n$ be distinct variables. Identifying $A\langle Z\rangle = A\langle Z_1, \dots, Z_n\rangle$ as usual with a subring of $A\langle V, Z\rangle$, this harmony means that for $f \in A\langle Z\rangle$ and $(v, z_1, \dots, z_n)$ in R^{n+1} we have $f(z_1, \dots, z_n) = f(v, z_1, \dots, z_n)$, where the last f refers to the image of the series $f \in A\langle Z\rangle$ in $A\langle V, Z\rangle$. Thus we can add dummy variables on the left. We can also add them at other places: identifying $f \in A\langle Z\rangle$ with its image in $A\langle Z_1, \dots, Z_i, V, Z_{i+1}, \dots, Z_n\rangle$ as usual, where $1 \leqslant i \leqslant n$, we have likewise

$$f(z_1, \dots, z_n) \ = \ f(z_1, \dots, z_i, v, z_{i+1}, \dots, z_n)$$

for $(z_1, \dots, z_i, v, z_{i+1}, \dots, z_n) \in R^{n+1}$. We shall tacitly use these facts.

Henselianity Again. Let R be an A-ring. Note that $\mathcal{O}(A)R$, that is, the ideal of R generated by $\iota_0\big(\mathcal{O}(A)\big)$, is contained in the Jacobson radical of R, because for $a_1, \dots, a_n \in \mathcal{O}(A)$ the series $1 + a_1 Y_1 + \cdots + a_n Y_n$ is a unit of $A\langle Y\rangle$. In later sections we consider the case that R is a valuation ring whose maximal ideal is $\sqrt{\mathcal{O}(A)R}$, and then the following is relevant:

Lemma 10.1. *The pair $(R, \mathcal{O}(A)R)$ is henselian, hence so is $(R, \sqrt{\mathcal{O}(A)R})$.*

Proof. Let $n = 1$, so $Y = Y_1$. We show that any polynomial

$$f(Y) \;=\; 1 + Y + z_2 Y^2 + \cdots + z_N Y^N \in R[Y], \qquad (N \in \mathbb{N}^{\geqslant 2})$$

with $z_2, \ldots, z_N \in \mathcal{o}(A)R$ has a zero in R. Take m and $x \in R^m$ such that

$$z_2 \;=\; g_2(x), \ldots, z_N \;=\; g_N(x), \quad g_2, \ldots, g_N \in \mathcal{o}(A)A[X] \subseteq \mathcal{o}(A)A\langle X\rangle.$$

Then $F(X,Y) := 1 + Y + g_2(X)Y^2 + \cdots + g_N(X)Y^N \in A[X,Y] = A\langle X, Y\rangle$ is regular in Y of degree 1, so $F(X,Y) = E \cdot (Y - c)$ for a unit E of $A\langle X, Y\rangle$ and $c \in A\langle X\rangle$. Thus $f(Y)$ has a zero $c(x)$ in R. □

Passing to A/I. Let R be an A-ring, and let (ι_n) be its A-analytic structure. Let I be a closed proper ideal of A contained in the kernel of ι_0. We equip A/I with its quotient norm, and observe that the ring morphism

$$A\langle Y\rangle \to (A/I)\langle Y\rangle, \quad f := \sum_\nu a_\nu Y^\nu \mapsto f/I := \sum_\nu (a_\nu + I)Y^\nu$$

is surjective and that its kernel contains $IA\langle Y\rangle$. Moreover, for $f \in IA\langle Y\rangle$ we have $f(y) = 0$ for all $y \in R^n$.

Lemma 10.2. *Suppose $a \in A$ with $a \notin A^\times$ and $\rho \in \mathbb{R}^>$ are such that $|ab| \geqslant \rho|b|$ for all $b \in A$. Then for $I := aA$ we have:*

(i) I is a closed proper ideal of A;

(ii) the kernel of the above morphism $A\langle Y\rangle \to (A/I)\langle Y\rangle$ is $IA\langle Y\rangle$;

(iii) $IA\langle Y\rangle$ is a closed proper ideal of $A\langle Y\rangle$, the induced ring isomorphism

$$A\langle Y\rangle/IA\langle Y\rangle \to (A/I)\langle Y\rangle$$

is norm preserving, with the quotient norm on $A\langle Y\rangle/IA\langle Y\rangle$;

(iv) we have an (A/I)-analytic structure $(\iota_n/I)_n$ on R given by

$$(\iota_n/I)(f/I) \;:=\; \iota_n(f) \;\; \textit{for } f \in A\langle Y\rangle.$$

Proof. This is routine. Towards verifying (A3) for $(\iota_n/I)_n$ one uses (iii) to show first that for $f, g_1, \ldots, g_n \in A\langle Y\rangle$ we have $(f/I)(g_1/I, \ldots, g_n/I) = f(g_1, \ldots, g_n)/I$. □

Passing to $A\langle t_p\rangle$ and $A[[t_p]]$. Let R be an A-ring. For $x \in R^m$ we equip R with the $A\langle X\rangle$-analytic structure $(\iota_{n,x})$ given for $f(X,Y) \in A\langle X\rangle\langle Y\rangle = A\langle X, Y\rangle$ by

$$\iota_{n,x} f \;:\; R^n \to R, \qquad y \mapsto f(x,y).$$

We refer to R with this $A\langle X\rangle$-analytic structure as *the (A,x)-ring R.*

We now combine Lemma 10.2 with the material on $A[t_p]$ in Section 9.3 as follows. Let $p(T) \in A[T]$ be monic of degree $d \geqslant 1$, and let $t \in R$ be such that $p(t) = 0$. Let us consider the (A,t)-ring R, that is, the ring R with the $A\langle T\rangle$-analytic structure $(\iota_{n,t})$. Note that $I := pA\langle T\rangle$ is a closed proper ideal of $A\langle T\rangle$ contained in the kernel of $\iota_{0,t}$, and $|pq| = |q|$ for all $q \in A\langle T\rangle$. Hence Lemma 10.2 yields an $A\langle t_p\rangle$-analytic structure $(\iota_{n,t}/I)$ on R. To simplify notation we set $\iota_{n,p} := \iota_{n,t}/I$. It is easy to check that $\iota_{0,p} : A\langle t_p\rangle \to R$ extends $\iota_0 : A \to R$, and $\iota_{0,p}(t_p) = t$. More generally, for $f_0, \dots, f_{d-1} \in A\langle Y\rangle$ and $y \in R^n$ we have

$$\begin{aligned}\iota_{n,p}\big(f_0 + t_p f_1 + \cdots + t_p^{d-1} f_{d-1}\big) &= \iota_n f_0 + t \cdot \iota_n f_1 + \cdots + t^{d-1} \cdot \iota_n f_{d-1}, \text{ so} \\ \big(f_0 + t_p f_1 + \cdots + t_p^{d-1} f_{d-1}\big)(y) &= f_0(y) + t f_1(y) + \cdots + t^{d-1} f_{d-1}(y).\end{aligned}$$

Next, assume also that $p = T^d + \sum_{i<d} p_i T^i$ with all $p_i \in \mathcal{O}(A)$. Then $(\iota_{n,p})$ is also an $A[[t_p]]$-analytic structure on R, since $A\langle t_p\rangle$ and $A[[t_p]]$ have the same underlying ring $A[t_p]$ and have equivalent norms as described in the subsection *Another norm on* $A[t_p]$ of Section 9.3.

Extensions of A-rings. Let R be an A-ring. When referring to an A-ring R^* as *extending* R this means of course that R is a subring of R^*, but also includes the requirement that the A-analytic structure of R^* extends that of R.

A set $S \subseteq R$ is said to be A-closed (in R) if for all m, $f \in A\langle X\rangle$ and $x_1, \dots, x_m$ in S we have $f(x_1, \dots, x_m) \in S$. Then S is a subring of R and the A-analytic structure of R restricts to an A-analytic structure on S. We view such S as an A-ring so as to make the A-ring R extend S. For $S \subseteq R$, the *A-closure of S in R* is the smallest (with respect to inclusion) A-closed subset of R that contains S.

Lemma 10.3. *Let R^* be an A-ring extending R, and $y = (y_1, \dots, y_n) \in (R^*)^n$. Let $R\langle y\rangle$ be the A-closure of $R \cup \{y_1, \dots, y_n\}$ in R^*. Then*

$$R\langle y\rangle \;=\; \bigcup_m \{g(x, y) : \; x \in R^m, \; g \in A\langle X, Y\rangle\}.$$

Here is a consequence of Lemma 9.16:

Lemma 10.4. *Suppose R^* is an A-ring that extends R. Let $f \in A\langle X, Y_1, \dots, Y_n\rangle$ and $x \in R^m$, and assume $y_1, \dots, y_n \in R^*$ are integral over R. Then*

$$f(x, y_1, \dots, y_n) \;\in\; R[y_1, \dots, y_n].$$

Proof. By increasing m and accordingly extending x with extra coordinates we arrange that for $j = 1, \dots, n$ we have a polynomial $f_j(X, Y_j) \in A[X, Y_j]$, monic in Y_j, with $f_j(x, y_j) = 0$. Now apply Lemma 9.16. □

Lemma 10.5. *Let R^* be a ring extension of R with $z \in R^*$ integral over R. Then at most one A-analytic structure on $R[z]$ makes $R[z]$ an A-ring extending R.*

Proof. We can assume $R^* = R[z]$. Take a monic polynomial $\phi \in R[Z]$, say of degree $d \geqslant 1$, with $\phi(z) = 0$. Let R^* be equipped with an A-analytic structure extending that of R, and let $g \in A\langle Y_1, \dots, Y_n\rangle$, $n \geqslant 1$, and let $y_1, \dots, y_n \in R^*$; we have to show that then the element $g(y_1, \dots, y_n) \in R^*$ does not depend on the given A-analytic structure on R^*. We have $\phi(Z) = x_{00} + x_{01}Z + \cdots + x_{0,d-1}Z^{d-1} + Z^d$ with $x_{00}, \dots, x_{0,d-1} \in R$ and $y_j = x_{j0} + x_{j1}z + \cdots + x_{j,d-1}z^{d-1}$, $x_{j0}, \dots, x_{j,d-1} \in R$, for $j = 1, \dots, n$. We now set $m := (1+n)d$ and

$$\begin{aligned} x &:= (x_{00}, \dots, x_{0,d-1}, x_{10}, \dots, x_{1,d-1}, \dots, x_{n0}, \dots, x_{n,d-1}) \in R^m, \\ X &= (X_1, \dots, X_m) := \big(X_{00}, \dots, X_{0,d-1}, \dots, X_{n0}, \dots, X_{n,d-1}\big), \end{aligned}$$

so $\phi(Z) = F(x,Z)$, $F(X,Z) := X_{00} + X_{01}Z + \cdots + X_{0,d-1}Z^{d-1} + Z^d \in A[X,Z]$. Let $G(X,Z) \in A\langle X,Z\rangle$ be the following substitution instance of g:

$$g\big(X_{10} + X_{11}Z + \cdots + X_{1,d-1}Z^{d-1}, \ldots, X_{n0} + X_{n1}Z + \cdots + X_{n,d-1}Z^{d-1}\big).$$

Lemma 9.15 gives $G(X,Z) = Q(X,Z)F(X,Z) + R_0 + R_1 Z + \cdots + R_{d-1}Z^{d-1}$ with $R_0, \ldots, R_{d-1} \in A\langle X\rangle$, and so $g(y) = G(x,z) = R_0(x) + R_1(x)z + \cdots + R_{d-1}(x)z^{d-1}$, which uses only the A-analytic structure on R, not that on R^*. $\square$

Proposition 10.6. *Let R^* be a ring extension of R and integral over R. Then some A-analytic structure on R^* makes R^* an A-ring extending R.*

Proof. In view of Lemmas 10.4 and 10.5 this reduces to the case $R^* = R[z]$ where $z \in R^*$ is integral over R. Let $\phi(Z) \in R[Z]$ be as in the proof of Lemma 10.5, in particular monic of degree $d \geqslant 1$ in Z. If the ring extension $R[Z]/\phi(Z)R[Z]$ of R can be given an A-analytic structure extending that of R, then this is also the case for its image $R[z]$ under the R-algebra morphism $R[Z] \to R[z]$ sending Z to z. Thus replacing $R[z]$ by $R[Z]/\phi(Z)R[Z]$ if necessary we arrange that $R[z]$ is free as an R-module with basis $1, z, \ldots, z^{d-1}$. We now adopt other notation from the proof above, where $n \geqslant 1$ and where we introduced a tuple

$$X \;=\; (X_1, \ldots, X_m) \;=\; (X_{00}, \ldots, X_{n,d-1})$$

of $m = (n+1)d$ distinct variables, the polynomial $F(X,Z) \in A[X,Z]$, and for any $g \in A\langle Y\rangle = A\langle Y_1, \ldots, Y_n\rangle$ the series $G = G(X,Z) \in A\langle X,Z\rangle$, and the series $R_0, \ldots, R_{d-1} \in A\langle X\rangle$. To indicate their dependence on g we set

$$\begin{aligned} G_g \;&:=\; G, \quad R_{g,0} \;:=\; R_0, \quad \ldots\;, \; R_{g,d-1} \;:=\; R_{d-1}, \\ R_g \;&:=\; R_{g,0} + R_{g,1}Z + \cdots + R_{g,d-1}Z^{d-1} \;\in\; A\langle X\rangle[Z]. \end{aligned}$$

We claim that setting $g(y) := R_g(x,z)$ for any $n \geqslant 1$ and $g \in A\langle Y\rangle$ yields an A-analytic structure on $R[z]$ extending that on R. We just verify two items that are part of this claim: let $f, g, h, g_1, \ldots, g_n \in A\langle Y\rangle$ and $y \in R^n$; then

1. $gh(y) \;=\; g(y)h(y)$;
2. $f(g_1, \ldots, g_n)(y) \;=\; f\big(g_1(y), \ldots, g_n(y)\big)$.

As to (1), we have $G_{gh} = G_g G_h$, so $R_{gh} \equiv R_g R_h \mod F$ in $A\langle X,Z\rangle$. We also have $R \in A\langle X\rangle[Z]$ of degree $< d$ in Z such that $R_g R_h \equiv R \mod F$ in $A\langle X\rangle[Z]$. Hence $R_{gh} = R$, and thus $gh(y) = R(x,z) = R_g(x,z)R_h(x,z) = g(y)h(y)$. As to (2), by Corollary 9.17 we have in $A\langle X,Z\rangle$,

$$G_{f(g_1,\ldots,g_n)} \;=\; f\big(G_{g_1}, \ldots, G_{g_n}\big) \;\equiv\; f(R_{g_1}, \ldots, R_{g_n}) \mod F.$$

Note that $f(R_{g_1}, \ldots, R_{g_n})$ is obtained by substituting $R_{g_j,i}$ for X_{ji} in G_f, for $j = 1, \ldots, n$ and $i = 0, \ldots, d-1$ (and Z for Z), that is,

$$f(R_{g_1}, \ldots, R_{g_n}) \;=\; G_f\big(R_{g_1,0}, \ldots, R_{g_1,d-1}, \ldots, R_{g_n,0}, \ldots, R_{g_n,d-1}, Z\big).$$

Making the same substitution in the congruence $G_f \equiv R_f \mod F$, using that the variables X_{ji} with $j = 1, \ldots, n$ and $i = 0, \ldots, d-1$ do not occur in F, we obtain

$$G_f\big(R_{g_1,0}, \ldots, R_{g_1,d-1}, \ldots, R_{g_n,0}, \ldots, R_{g_n,d-1}, Z\big)$$

is congruent in $A\langle X, Z\rangle$ modulo F to

$$R_f(X_{00}, \ldots, X_{0,d-1}, R_{g_1,0}, \ldots, R_{g_1,d-1}, \ldots, R_{g_n,0}, \ldots, R_{g_n,d-1}, Z\big),$$

which is in $A\langle X\rangle[Z]$ of degree $< d$ in Z, and thus equals $R_{f(g_1,\ldots,g_n)}$. Since

$$f\big(g_1(y), \ldots, g_n(y)\big) \ = \ R_f\big(x_{00}, \ldots, x_{0,d-1}, R_{g_1,0}(x), \ldots, R_{g_n,d-1}(x), z\big),$$

this yields $f\big(g_1(y), \ldots, g_n(y)\big) = f(g_1, \ldots, g_n)(y)$, as required. □

Corollary 10.7. *If R^* is a ring extension of R and integral over R, then there is a unique A-analytic structure on R^* that makes R^* an A-ring extending R.*

Corollary 10.8. *Let R_1 and R_2 be A-rings extending R and let $\phi : R_1 \to R_2$ be an R-algebra morphism such that $\phi(R_1)$ is integral over R. Then ϕ is a morphism of A-rings (that is, a homomorphism in the sense of $\mathcal{L}^A$-structures).*

Proof. The kernel of ϕ is a congruence relation on R as A-ring, so $\phi(R_1)$ has an A-analytic structure making $\phi : R_1 \to \phi(R_1)$ a morphism of A-rings. Since $\phi(R_1)$ is A-closed as a subset of R_2 it follows from Corollary 10.7 that this A-analytic structure on $\phi(R_1)$ coincides with the one that makes the inclusion $\phi(R_1) \to R_2$ a morphism of A-rings. Thus ϕ is a morphism of A-rings. □

Corollary 10.9. *Suppose the A-ring R^* extends R, and $z_i \in R^*$ for $i \in I$ is integral over R. Then $R[z_i : i \in I]$ is A-closed in R^*.*

Proof. Let $f(Y) \in A\langle Y\rangle$ and suppose $y_1, \ldots, y_n \in R^*$ are integral over R; it suffices to show that then $f(y) \in R[y]$ where $y = (y_1, \ldots, y_n)$. Take $x \in R^m$ and monic $f_j \in A\langle X\rangle[Y_j]$ such that $f_j(x, y_j) = 0$ for $j = 0, \ldots, n$, and apply Lemma 9.16. □

Defining $R\langle Y\rangle$. Let R be an A-ring. To define a ring $R\langle Y\rangle$ analogous to the polynomial ring $R[Y]$, observe that polynomials over R arise from polynomials over $\mathbb{Z}$ by specializing: for $f(X, Y) \in \mathbb{Z}[X, Y]$ and $x \in R^m$ we have $f(x, Y) \in R[Y]$. We take this as a hint and with A instead of $\mathbb{Z}$, we define for $f(X, Y) \in A\langle X, Y\rangle$ and $x \in R^m$ the power series $f(x, Y) \in R[[Y]]$ as follows: $f(X, Y) = \sum_\nu f_\nu(X)Y^\nu$ with all $f_\nu \in A\langle X\rangle$, and then

$$f(x, Y) \ := \ \sum_\nu f_\nu(x)Y^\nu.$$

Thus for fixed $x \in R^m$ the map $f(X, Y) \mapsto f(x, Y) : A\langle X, Y\rangle \to R[[Y]]$ is an A-algebra morphism. We define

$$R\langle Y\rangle \ := \ \bigcup_m \{f(x, Y) : \ f = f(X, Y) \in A\langle X, Y\rangle, \ x \in R^m\} \ \subseteq \ R[[Y]].$$

An easy consequence is that inside the ambient ring $R[[Y]]$ we have for $i \leqslant n$:

$$R\langle Y_1, \ldots, Y_i\rangle \ = \ R\langle Y\rangle \cap R[[Y_1, \ldots, Y_i]].$$

Lemma 10.10. *Given any $g_1, \ldots, g_k \in R\langle Y \rangle$, $k \in \mathbb{N}$, there exists m, $x \in R^m$, and $f_1, \ldots, f_k \in A\langle X, Y \rangle$, such that $g_1 = f_1(x, Y), \ldots, g_k = f_k(x, Y)$.*

Proof. Let $m_1, \ldots, m_k \in \mathbb{N}$ and $x^1 \in R^{m_1}, \ldots, x^k \in R^{m_k}$ be such that

$$\begin{aligned} g_1 &= f^1(x^1, Y), \ldots, g_k = f^k(x^k, Y), \quad f^1 \in A\langle X^1, Y \rangle, \ldots, f^k \in A\langle X^k, Y \rangle, \\ x^1 &= (x_{11}, \ldots, x_{1m_1}), \ \ldots, \ x^k = (x_{k1}, \ldots, x_{km_k}), \\ X^1 :&= (X_{11}, \ldots, X_{1m_1}), \ \ldots, \ X^k := (X_{k1}, \ldots, X_{km_k}). \end{aligned}$$

We can also arrange for $m := m_1 + \cdots + m_k$ that

$$X = (X_1, \ldots, X_m) = (X^1, \ldots, X^k).$$

For $f_1(X, Y) := f^1(X^1, Y) \in A\langle X, Y \rangle, \ldots, f_k(X, Y) := f^k(X^k, Y) \in A\langle X, Y \rangle$ we then have $f_1(x, Y) = g_1, \ldots, f_k(x, Y) = g_k$ for $x = (x^1, \ldots, x^k) \in R^m$. □

Corollary 10.11. *$R\langle Y \rangle$ is a subring of $R[[Y]]$ with $R[Y] \subseteq R\langle Y \rangle$. If R is a domain, then so is $R\langle Y \rangle$; if R has no nilpotents other than 0, then neither does $R\langle Y \rangle$. For an A-ring R^* extending R the inclusion $R[[Y]] \to R^*[[Y]]$ maps $R\langle Y \rangle$ into $R^*\langle Y \rangle$, so $R\langle Y \rangle$ is a subring of $R^*\langle Y \rangle$.*

Proof. The claim about domains holds because it holds with $R[[Y]]$ in place of $R\langle Y \rangle$. Suppose R has no nilpotents. With $\mathfrak{p}$ ranging over the prime ideals of R this yields an injective "diagonal" ring morphism $R[[Y]] \to \prod_{\mathfrak{p}} (R/\mathfrak{p})[[Y]]$ into a ring with no nilpotents other than 0, so $R[[Y]]$ has no such nilpotents either. □

By the remark following the definition of $R\langle Y \rangle$ we have for $i \leqslant n$ the subring $R\langle Y_1, \ldots, Y_i \rangle$ of $R\langle Y \rangle$. The ring $A\langle Y \rangle$ as defined in Section 9.3 is the same as the ring $A\langle Y \rangle$ as defined just now for $R = A$ viewed as an A-ring.

Corollary 10.12. *Suppose the A-ring R^* extends R and is integral over R. Then $R^*\langle Y \rangle$ is generated as a ring over its subring $R\langle Y \rangle$ by R^*.*

Proof. Using Corollary 10.9 it suffices to consider the case $R^* = R[z]$ where $z \in R^*$ is integral over R and to show $R^*\langle Y \rangle = R\langle Y \rangle[z]$. Let $f(X, Y) \in A\langle X, Y \rangle$ and $x \in (R^*)^m$. Towards proving $f(x, Y) \in R\langle Y \rangle[z]$, let $\phi(z) = 0$ where

$$\phi(Z) = Z^d + u_{0,d-1} Z^{d-1} + \cdots + u_{0,0}, \quad d \geqslant 1, \quad u_{0,0}, \ldots, u_{0,d-1} \in R.$$

Then for $i = 1, \ldots, m$ we have $x_i = u_{i0} + u_{i1} z + \cdots + u_{i,d-1} z^{d-1}$ with all $u_{ij} \in R$. Let $U = (U_{ij})_{0 \leqslant i \leqslant m, j < d}$ be a tuple of distinct variables different also from the Y_j and Z and set $u := (u_{ij})$. Then $f(x, Y) = g(u, z, Y)$ where

$$g(U, Z, Y) = f\Big(\sum_{j<d} U_{1j} Z^i, \ldots, \sum_{j<d} U_{mj} Z^i, Y\Big) \in A\langle U, Z, Y \rangle.$$

In the ring $A\langle U, Z, Y \rangle$, $g(U, Z, Y)$ is congruent modulo $Z^d + U_{0,d-1} Z^{d-1} + \cdots + U_{0,0}$ to $g_0(U, Y) + g_1(U, Y) Z + \cdots + g_{d-1}(U, Y) Z^{d-1}$ for suitable $g_0, \ldots, g_{d-1} \in A\langle U, Y \rangle$, by Lemma 9.15, and for such g_j we have

$$g(u, z, Y) = g_0(u, Y) + g_1(u, Y) z + \cdots + g_{d-1}(u, Y) z^{d-1} \in R\langle Y \rangle[z]. \qquad \square$$

Construing R for $x \in R^m$ as an (A, x)-ring gives the same subring $R\langle Y \rangle$ of $R[[Y]]$ as when considering R as an A-ring.

Let $p(T) \in A[T]$ be monic of degree $d \geqslant 1$ with $p(t) = 0$, $t \in R$. Then we may construe R as an $A\langle t_p \rangle$-ring, and this gives again the same subring $R\langle Y \rangle$ of $R[[Y]]$. If in addition $p = T^d + \sum_{i<d} p_i T^i$ with all $p_i \in \mathcal{O}(A)$ and R is viewed accordingly as an $A[[t_p]]$-ring, then this also yields the same $R\langle Y \rangle$.

$R\langle Z \rangle$ as an A-ring. Let $Z_1, \dots, Z_N$ with $N \in \mathbb{N}$ be distinct variables different from $X_1, X_2, \dots$, and set $Z := (Z_1, \dots, Z_N)$. We define $R\langle Z \rangle = R\langle Z_1, \dots, Z_N \rangle$ in the same way as $R\langle Y_1, \dots, Y_N \rangle$, with $Z_1, \dots, Z_N$ in the role of $Y_1, \dots, Y_N$. We make $R\langle Z \rangle$ an A-ring extending R as follows. Let $f \in A\langle Y \rangle$ and $u_1(x, Z), \dots, u_n(x, Z)$ in $R\langle Z \rangle$, where $u_1(X, Z), \dots, u_n(X, Z) \in A\langle X, Z \rangle$ and $x \in R^m$. Set

$$g(X, Z) \ := \ f\big(u_1(X, Z), \dots, u_n(X, Z)\big) \in A\langle X, Z \rangle.$$

Our aim is to define $f(u_1(x, Z), \dots, u_n(x, Z)) := g(x, Z) \in R\langle Z \rangle$. In order for this to make sense as a definition we first show:

Lemma 10.13. *Suppose $v_j(X, Z) \in A\langle X, Z \rangle$ and $u_j(x, Z) = v_j(x, Z)$ for $j = 1, \dots, n$. Set $h(X, Z) := f\big(v_1(X, Z), \dots, v_n(X, Z)\big) \in A\langle X, Z \rangle$. Then*

$$g(x, Z) \ = \ h(x, Z).$$

Proof. By Corollary 9.17 we have for distinct variables $U_1, \dots, U_n, V_1, \dots, V_n$,

$$f(U_1, \dots, U_n) - f(V_1, \dots, V_n) \in (U_1 - V_1, \dots, U_n - V_n)A\langle U, V \rangle.$$

Substituting $u_j(X, Z)$ and $v_j(X, Z)$ for U_j and V_j gives

$$g(X, Z) - h(X, Z) \in \big(u_1(X, Z) - v_1(X, Z), \dots, u_n(X, Z) - v_n(X, Z)\big)A\langle X, Z \rangle$$

from which we obtain the desired result by substituting x for X. $\square$

Now the next lemma shows the above does define $f\big(u_1(x, Z), \dots, u_n(x, Z)\big)$:

Lemma 10.14. *Let $m_1, m_2 \in \mathbb{N}$, $m := m_1 + m_2$, and*

$$\begin{aligned} X^1 \ &:= \ (X_1, \dots, X_{m_1}), \quad X^2 \ := \ (X_{m_1+1}, \dots, X_m), \\ x^1 \ &= \ (x_1, \dots, x_{m_1}) \in R^{m_1}, \quad x^2 \ = \ (x_{m_1+1}, \dots, x_m) \in R^{m_2}. \end{aligned}$$

Suppose that the series

$$u^1(X^1, Z), \dots, u^n(X^1, Z) \in A\langle X^1, Z \rangle, \quad v^1(X^2, Z), \dots, v^n(X^2, Z) \in A\langle X^2, Z \rangle$$

are such that $u^1(x^1, Z) = v^1(x^2, Z), \dots, u^n(x^1, Z) = v^n(x^2, Z)$. Then for

$$\begin{aligned} g^1(X^1, Z) \ &:= \ f\big(u^1(X^1, Z), \dots, u^n(X^1, Z)\big) \in A\langle X^1, Z \rangle, \\ g^2(X^2, Z) \ &:= \ f\big(v^1(X^2, Z), \dots, v^n(X^2, Z)\big) \in A\langle X^2, Z \rangle \end{aligned}$$

we have $g^1(x^1,Z)=g^2(x^2,Z)$ *in* $R\langle Z\rangle$.

Proof. Set $X:=(X^1,X^2)=(X_1,\dots,X_m)$, and for $j=1,\dots,n$,

$$\begin{aligned} u_j(X,Z) &:= u^j(X^1,Z)\in A\langle X,Z\rangle, \quad v_j(X,Z) := v^j(X^2,Z)\in A\langle X,Z\rangle,\\ g(X,Z) &:= f\big(u_1(X,Z),\dots,u_n(X,Z)\big) = g^1(X^1,Z)\in A\langle X,Z\rangle,\\ h(X,Z) &:= f\big(v_1(X,Z),\dots,v_n(X,Z)\big) = g^2(X^2,Z)\in A\langle X,Z\rangle. \end{aligned}$$

Then for $x=(x_1,\dots,x_m)$ we have $u_j(x,Z)=v_j(x,Z)$ for $j=1,\dots,n$, so $g(x,Z)=h(x,Z)$ by Lemma 10.13, and thus $g^1(x^1,Z)=g^2(x^2,Z)$. $\square$

We have now defined for $f\in A\langle Y\rangle$ a corresponding operation

$$(u_1,\dots,u_n)\mapsto f(u_1,\dots,u_n)\ :\ R\langle Z\rangle^n\to R\langle Z\rangle.$$

This makes $R\langle Z\rangle$ an A-ring extending R. For $f\in A\langle X,Z\rangle$ and $x\in R^m$ we can interpret $f(x,Z)$ on the one hand as the element of $R[[Z]]$ defined in the beginning of this subsection (with Y instead of Z), but also as the element of $R\langle Z\rangle$ obtained by evaluating f at the point $(x,Z)\in R\langle Z\rangle^{m+N}$ according to the A-analytic structure we gave $R\langle Z\rangle$; one checks easily that these two interpretations give the same element of $R\langle Z\rangle$, so there is no conflict of notation. This also shows that $R\langle Z\rangle$ is generated as an A-ring by its subset $R\cup\{Z_1,\dots,Z_N\}$.

For $R=A$ as an A-ring the above yields the A-ring $A\langle Z\rangle$ extending A. Let $f=f(Y)=\sum_\nu a_\nu Y^\nu\in A\langle Y\rangle$. One checks easily that for $(g_1,\dots,g_n)\in A\langle Z\rangle^n$ the convergent sum $f(g_1,\dots,g_n)=\sum_\nu a_\nu g_1^{\nu_1}\cdots g_n^{\nu_n}\in A\langle Z\rangle$ equals $f(g_1,\dots,g_n)$ as defined above for $R=A$, so this causes no conflict of notation. It is routine to check that for any A-ring R and $z\in R^N$ the evaluation map $g\mapsto g(z): A\langle Z\rangle\to R$ is a morphism of A-rings. For $N=0$ this is just $\iota_0: A\to R$.

Corollary 10.15. *Let* $J:=\sqrt{\mathcal{O}(A)R}$. *Then* $\big(R\langle Z\rangle, JR\langle Z\rangle\big)$ *is henselian.*

Proof. Applying Lemma 10.1 to the A-ring $R\langle Z\rangle$, the pair $\big(R\langle Z\rangle,\sqrt{\mathcal{O}(A)R\langle Z\rangle}\big)$ is henselian. Now use that $JR\langle Z\rangle\subseteq\sqrt{\mathcal{O}(A)R\langle Z\rangle}$. $\square$

Let $p(T)=T^d+\sum_{i<d}p_iT^i\in A[T]$ with $d\geqslant 1$ and all $p_i\in\mathcal{O}(A)$, and let $t\in R$ be such that $p(t)=0$. We expand the A-ring R accordingly to an $A[[t_p]]$-ring such that the image of t_p in R is t, as described in the subsection *Passing to* $A\langle t_p\rangle$ *and* $A[[t_p]]$. This makes $R\langle Z\rangle$ an $A[[t_p]]$-ring, and as such it expands the A-ring $R\langle Z\rangle$.

Our next goal is to define for $z\in R^N$ an evaluation map $g\mapsto g(z): R\langle Z\rangle\to R$. We do this in the next section under a further noetherian assumption on A.

10.2 The case of noetherian A

Let A be a noetherian ring with an ideal $\mathcal{O}(A)\neq A$ such that $\bigcap_e\mathcal{O}(A)^e=\{0\}$ (with e ranging here and below over $\mathbb{N}$) and A is $\mathcal{O}(A)$-adically complete. Taking $0<\delta<1$ and defining $|a|:=\delta^n$ if $a\in\mathcal{O}(A)^n\setminus\mathcal{O}(A)^{n+1}$ for $a\in A^{\neq}$ and $|0|:=0$ gives an ultranorm on A with respect to which A is complete, with $\mathcal{O}(A)=\{a\in A:\ |a|<1\}$. Then the $\mathcal{O}(A)$-adic topology is the norm-topology. Take $t_1,\dots,t_r\in A$, $r\in\mathbb{N}$, such that $\mathcal{O}(A)=(t_1,\dots,t_r)$. Below, $n\geqslant 1$ and $Y=(Y_1,\dots,Y_n)$ as before, and λ,μ,ν range over $\mathbb{N}^n$.

Lemma 10.16. *Let $f = \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$. Then there is $d \in \mathbb{N}^{\geqslant 1}$ such that for all ν with $|\nu| \geqslant d$ we have $a_\nu = \sum_{|\mu|<d} a_\mu b_{\mu\nu}$ where the $b_{\mu\nu} \in \mathcal{o}(A)$ can be chosen such that $b_{\mu\nu} \to 0$ as $|\nu| \to \infty$ for each fixed μ with $|\mu| < d$.*

Proof. Since $a_\nu \to 0$ as $|\nu| \to \infty$, we have $a_\nu \in \mathcal{o}(A)^{e(\nu)}$ with $e(\nu) \in \mathbb{N}$, $e(\nu) \to \infty$ as $|\nu| \to \infty$. So $a_\nu = P_\nu(t_1, \dots, t_r)$ with $P_\nu \in A[T_1, \dots, T_r]$ homogeneous of degree $e(\nu)$. Take $d_0 \in \mathbb{N}$ such that the ideal of $A[T_1, \dots, T_r]$ generated by the P_ν is already generated by the P_μ with $|\mu| < d_0$. Next take $d \geqslant d_0$ in $\mathbb{N}^{\geqslant 1}$ so large that $e(\nu) > e(\mu)$ for all μ, ν with $|\mu| < d_0$ and $|\nu| \geqslant d$. Let $|\nu| \geqslant d$. Then $P_\nu = \sum_{|\mu|<d_0} P_\mu Q_{\mu\nu}$ with each $Q_{\mu\nu} \in A[T_1, \dots, T_r]$ homogeneous of degree $e(\nu) - e(\mu)$. Hence

$$a_\nu \;=\; \sum_{|\mu|<d_0} a_\mu b_{\mu\nu}, \qquad b_{\mu\nu} \;:=\; Q_{\mu\nu}(t_1, \dots, t_r) \in \mathcal{o}(A),$$

which yields the desired result. $\square$

Let f, d, and the $b_{\mu\nu}$ be as in the lemma. For μ with $|\mu| < d$ we set

$$f_\mu \;:=\; Y^\mu + \sum_{|\nu|\geqslant d} b_{\mu\nu} Y^\nu \in A\langle Y\rangle, \quad \text{so} \quad f \;=\; \sum_{|\mu|<d} a_\mu f_\mu.$$

Therefore $\mathcal{o}(A\langle Y\rangle) = (t_1, \dots, t_r)A\langle Y\rangle$ and the ultranorm on $A\langle Y\rangle$ induced by the above ultranorm on A has the property that for all $f \in A\langle Y\rangle$ and n,

$$|f| \leqslant \delta^n \iff f \in \mathcal{o}(A\langle Y\rangle)^n,$$

so the norm-topology of $A\langle Y\rangle$ is the same as its $\mathcal{o}(A\langle Y\rangle)$-adic topology. Moreover, for all $f \in A[Y]$ and n,

$$f \in \mathcal{o}(A\langle Y\rangle)^n \iff f \in \mathcal{o}(A)^n A[Y],$$

so $A\langle Y\rangle$ is noetherian by [46, Theorem 8.12]. Thus $A\langle Y\rangle$ inherits the conditions we imposed on A at the beginning of this section.

In the rest of this section R is an A-ring, and $Z = (Z_1, \dots, Z_N)$ as before.

Lemma 10.17. *Let $f = \sum_\nu a_\nu(X) Y^\nu \in A\langle X\rangle\langle Y\rangle = A\langle X, Y\rangle$. Suppose $x \in R^m$ and $f(x, Y) = 0$, that is, $a_\nu(x) = 0$ for all ν. Then $f(x, y) = 0$ for all $y \in R^n$.*

Proof. With $A\langle X\rangle$ in the role of A, the above gives a finite sum decomposition $f = \sum_{|\mu|<d} a_\mu f_\mu$ with the $f_\mu \in A\langle X, Y\rangle$, which yields the desired conclusion. $\square$

We can now prove the following key universal property of the A-ring $R\langle Z\rangle$:

Theorem 10.18. *Let $\phi : R \to R^*$ be an A-ring morphism and $z = (z_1, \dots, z_N) \in (R^*)^N$. Then ϕ extends uniquely to an A-ring morphism $R\langle Z\rangle \to R^*$ sending $Z_1, \dots, Z_N$ to $z_1, \dots, z_N$, respectively.*

Proof. Let $g(Z) \in R\langle Z\rangle$. Take $f(X, Z) \in A\langle X, Z\rangle$ and $x \in R^m$ such that $g(Z) = f(x, Z)$, and set $g(z) := f(\phi(x), z) \in R^*$. By Lemma 10.17 (with Z instead of Y) and the usual arguments with dummy variables, this element of R^* depends only on $g(Z)$ and z, not on the choice of m, f, x. Moreover, the map $g(Z) \mapsto g(z) : R\langle Z\rangle \to R^*$ is a ring morphism that extends ϕ and sends Z_j to z_j for $j = 1, \dots, N$. One also verifies easily that for $F \in A\langle Y\rangle$ and $g_1, \dots, g_n \in R\langle Z\rangle$ we have

$$F(g_1, \dots, g_n)(z) \ = \ F\big(g_1(z), \dots, g_n(z)\big),$$

so this map $R\langle Z\rangle \to R^*$ is an A-ring morphism. $\square$

We retain the notation $g(z)$ introduced in the proof above. In Theorem 10.18, $g \in R\langle Z_1, \dots, Z_i\rangle$ with $i \leqslant N$ gives $g(z_1, \dots, z_i) = g(z_1, \dots, z_N)$ where on the right we take g as an element of $R\langle Z\rangle$. For $R = R^*$ and ϕ the identity on R this theorem gives the evaluation map $g \mapsto g(z) : R\langle Z\rangle \to R$ promised earlier as a morphism of A-rings. It is also a morphism of R-algebras.

Lemma 10.19. *Let $z \in R^N$. Then the kernel of the morphism $g \mapsto g(z) : R\langle Z\rangle \to R$ of R-algebras is the ideal $(Z_1 - z_1, \dots, Z_N - z_N)R\langle Z\rangle$ of $R\langle Z\rangle$.*

Proof. For $N \geqslant 1$, Lemma 9.15 gives $R\langle Z\rangle = (Z_N - z_N)R\langle Z\rangle + R\langle Z_1, \dots, Z_{N-1}\rangle$. Proceeding inductively we obtain $R\langle Z\rangle = (Z_1 - z_1, \dots, Z_N - z_N)R\langle Z\rangle + R$, which gives the desired result. $\square$

Note also that the map

$$R\langle Z\rangle \to \text{ring of } R\text{-valued functions on } R^N$$

assigning to each $g \in R\langle Z\rangle$ the function $z \mapsto g(z)$ is an R-algebra morphism.

Another special case of Theorem 10.18: let R^* be an A-ring extending R, let ϕ be the resulting inclusion $R \to R^*\langle Z\rangle$, and $z_j := Z_j \in R^*\langle Z\rangle$ for $j = 1, \dots, N$. Then the corresponding A-ring morphism $R\langle Z\rangle \to R^*\langle Z\rangle$ is a restriction of the inclusion $R[[Z]] \to R^*[[Z]]$ and sends $f(x, Z) \in R\langle Z\rangle$ for $f \in A\langle X, Z\rangle$ and $x \in R^m$ to $f(x, Z) \in R^*\langle Z\rangle$. We identify $R\langle Z\rangle$ with an A-subring of $R^*\langle Z\rangle$ via this morphism. Thus in the situation of Lemma 10.3 we have

$$R\langle y\rangle \ = \ \{g(y) : \ g \in R\langle Y\rangle\}.$$

Let I be an ideal of R. Then the canonical map $R \to R/I$ extends to the morphism $R\langle Z\rangle \to (R/I)\langle Z\rangle$ of A-rings sending Z_j to Z_j for $j = 1, \dots, N$, and we have:

Lemma 10.20. *The kernel of the above morphism $R\langle Z\rangle \to (R/I)\langle Z\rangle$ is $IR\langle Z\rangle$.*

Proof. With β ranging over $\mathbb{N}^N$, this morphism is a restriction of the ring morphism

$$\sum_\beta c_\beta Z^\beta \mapsto \sum_\alpha (c_\beta + I)Z^\beta \ : \ R[[Z]] \to (R/I)[[Z]] \qquad (\text{ all } c_\beta \in R),$$

so $IR\langle Z\rangle$ is contained in the kernel. Suppose $f(x, Z)$ is in the kernel where $x \in R^m$ and $f(X, Z) = \sum_\beta a_\beta(X)Z^\beta \in A\langle X, Z\rangle$. Then all $a_\beta(x) \in I$, and since for some $d \geqslant 1$ we have an equality $f(X, Z) = \sum_{|\alpha|<d} a_\alpha(X) f_\alpha(X, Z)$ with α ranging over $\mathbb{N}^N$ and all $f_\alpha(X, Z) \in A\langle X, Z\rangle$, substitution of x for X gives $f(x, Z) \in IR\langle Z\rangle$. $\square$

Corollary 10.21. *Let $J := \sqrt{\mathcal{o}(A)R}$. Then $JR\langle Z\rangle = \sqrt{\mathcal{o}(A)R\langle Z\rangle}$.*

Proof. We have $\mathcal{o}(A)R\langle Z\rangle \subseteq JR\langle Z\rangle \subseteq \sqrt{\mathcal{o}(A)R\langle Z\rangle}$. It remains to note that $JR\langle Z\rangle$ is a radical ideal of $R\langle Z\rangle$, by Lemma 10.20 and a part of Corollary 10.11. $\square$

Let $x \in R^m$, construe R as an (A,x)-ring, so R is equipped with a certain $A\langle X\rangle$-analytic structure, and let $\phi : R \to R^*$ be an A-ring morphism. Then $\phi : R \to R^*$ is also an $A\langle X\rangle$-ring morphism where we construe R^* as an $(A, \phi(x))$-ring, with $\phi(x) := \big(\phi(x_1), \dots, \phi(x_m)\big)$. For $z \in (R^*)^N$ the unique extension of ϕ to an A-ring morphism $R\langle Z\rangle \to R^*$ sending $Z_1, \dots, Z_N$ to $z_1, \dots, z_N$ is also an $A\langle X\rangle$-ring morphism. In other words, for $g \in R\langle Z\rangle$ and $z \in (R^*)^N$ the two ways of interpreting $g(z)$ give the same element of R^*, and so this raises no conflict of notation.

Next, let $p(T) = T^d + \sum_{i<d} p_i T^i \in A[T]$ with $d \geqslant 1$ and all $p_i \in \mathcal{O}(A)$. The following is a routine consequence of $\|t_p\| < 1$ and Lemma 9.14:

Lemma 10.22. *The ring $A[t_p]$ is noetherian, and $\bigcap_e \mathcal{O}(A[[t_p]])^e = \{0\}$. The topology on $A[[t_p]]$ given by its norm $\|\cdot\|$ equals its $\mathcal{O}(A[[t_p]])$-adic topology.*

Thus $A[[t_p]]$ inherits the properties that we imposed on A in the beginning of this section. Suppose now also that $p(t) = 0$, $t \in R$. We expand the A-ring R accordingly to an $A[[t_p]]$-ring such that the image of t_p in R is t, as described in *Passing to $A\langle t_p\rangle$ and $A[[t_p]]$*. As in Theorem 10.18, let $\phi : R \to R^*$ be an A-ring morphism. Then $p\big(\phi(t)\big) = 0$, and we accordingly expand the A-ring R^* to an $A[[t_p]]$-ring such that the image of t_p is $\phi(t)$. Then $\phi : R \to R^*$ is an $A[[t_p]]$-ring morphism and for $z \in (R^*)^N$ its unique extension to an A-ring morphism $R\langle Z\rangle \to R^*$ sending $Z_1, \dots, Z_N$ to $z_1, \dots, z_N$ is also an $A[[t_p]]$-ring morphism. In other words, for $g \in R\langle Z\rangle$ and $z \in (R^*)^N$ the two ways of interpreting $g(z)$ give the same element of R^*, and so this raises no conflict of notation.

Substituting elements of $R\langle Z\rangle$ in elements of $R\langle Y\rangle$. Here is another case of Theorem 10.18: let $g_1, \dots, g_n \in R\langle Z\rangle$ and $\phi : R \to R\langle Z\rangle$ the inclusion map. Then ϕ extends uniquely to the A-ring morphism

$$f \mapsto f(g_1, \dots, g_n) \ : \ R\langle Y\rangle \to R\langle Z\rangle$$

that sends $Y_1, \dots, Y_n$ to $g_1, \dots, g_n$. For $z \in R^N$ we have

$$f(g_1, \dots, g_n)(z) \ = \ f\big(g_1(z), \dots, g_n(z)\big).$$

This follows for example from the uniqueness in Theorem 10.18.

Let now $n, d \geqslant 1$. For $N = n$ and $Y = Z$ this yields the automorphism

$$f(Y) \mapsto f\big(T_d(Y)\big)$$

of the A-ring $R\langle Y\rangle$ and the R-algebra $R\langle Y\rangle$, with inverse $g(Y) \mapsto g\big(T_d^{-1}(Y)\big)$.

Let $f \in A\langle X, Z\rangle$, $g_1(X,Z), \dots, g_N(X,Z) \in A\langle X, Z\rangle$, and set

$$h(X,Z) \ := \ f\big(X, g_1(X,Z), \dots, g_N(X,Z)\big) \in A\langle X, Z\rangle.$$

Then for $x \in R^m$ we can interpret $f\big(x, g_1(x,Z), \dots, g_N(x,Z)\big)$ on the one hand as $h(x,Z) \in R\langle Z\rangle$, and on the other hand as the element of $R\langle Z\rangle$ obtained by evaluating f at the point $\big(x, g_1(x,Z), \dots, g_N(x,Z)\big) \in R\langle Z\rangle^{m+N}$ according to the A-analytic structure we gave $R\langle Z\rangle$. By the uniqueness in Theorem 10.18 these two elements of $R\langle Z\rangle$ are equal, so this raises no conflict of notation.

Introducing $K\langle Y\rangle$. Let the A-ring R be a domain with fraction field K. Set

$$K\langle Y\rangle \ := \ \{c^{-1}g(Y) : \ c \in R^{\neq},\ g(Y) \in R\langle Y\rangle \subseteq K[[Y]]\}.$$

Thus $K\langle Y\rangle$ is a subring of $K[[Y]]$ and contains $R\langle Y\rangle$ as a subring. For $n, d \geqslant 1$ the automorphism $g(Y) \mapsto g\big(T_d(Y)\big)$ of the A-ring $R\langle Y\rangle$ extends (uniquely) to an automorphism of the K-algebra $K\langle Y\rangle$, also to be indicated by $g \mapsto g\big(T_d(Y)\big)$.

Lemma 10.23. *$K\langle Y\rangle \cap R[[Y]] = R\langle Y\rangle$, inside the ambient ring $K[[Y]]$.*

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, let $g \in K\langle Y\rangle \cap R[[Y]]$. Now $g = c^{-1}\sum_\nu a_\nu(x)Y^\nu$ with $c \in R^{\neq}$ and $\sum_\nu a_\nu(X)Y^\nu \in A\langle X, Y\rangle$, $x \in R^m$. By Lemma 10.16 applied to $A\langle X\rangle$ instead of A we have $d \in \mathbb{N}^{\geqslant 1}$ such that for all ν with $|\nu| \geqslant d$ we have $a_\nu(X) = \sum_{|\mu|<d} a_\mu(X)b_{\mu\nu}(X)$ where the $b_{\mu\nu} \in \sigma(A\langle X\rangle)$ are chosen such that $b_{\mu\nu} \to 0$ as $|\nu| \to \infty$ for each fixed μ with $|\mu| < d$. Put $u_\mu := c^{-1}a_\mu(x) \in R$ for $|\mu| < d$. Then with a tuple $U = (U_\mu)_{|\mu|<d}$ of new variables and setting

$$F(X, U, Y) \ := \ \sum_{|\mu|<d} U_\mu Y^\nu + \sum_{|\nu|\geqslant d}\big(\sum_{|\mu|<d} b_{\mu\nu}(X)U_\mu\big)Y^\nu \in A\langle X, U, Y\rangle$$

we have $g(Y) = F(x, u, Y) \in R\langle Y\rangle$. $\square$

For $y \in R^n$ and $f(Y) = c^{-1}g(Y) \in K\langle Y\rangle$ with $c \in R^{\neq}, g(Y) \in R\langle Y\rangle$, the element $c^{-1}g(y) \in K$ depends only on f, y, not on c, g, and so we can define $f(y) := c^{-1}g(y)$. The map $f \mapsto f(y) : K\langle Y\rangle \to K$ is a K-algebra morphism, extending the evaluation maps $K[Y] \to K$ and $R[Y] \to R$ sending $Y_1, \ldots, Y_n$ to $y_1, \ldots, y_n$, respectively. By Lemma 10.19 its kernel is the maximal ideal $(Y_1 - y_1, \ldots, Y_n - y_n)K\langle Y\rangle$ of $K\langle Y\rangle$.

Suppose the A-ring S extends R, and is a domain with fraction field L taken as a field extension of K. Then $L[[Y]]$ has subrings $R\langle Y\rangle,\ K\langle Y\rangle,\ S\langle Y\rangle,\ L\langle Y\rangle$ with $K\langle Y\rangle \subseteq L\langle Y\rangle$. In this situation we have:

Lemma 10.24. *Assume S is integral over R and $b_1, \ldots, b_m$ is a basis of the K-linear space L. Then $L\langle Y\rangle$ is a free $K\langle Y\rangle$-module with basis $b_1, \ldots, b_m$.*

Proof. Let $g \in L\langle Y\rangle$ and take $c \in L^\times$ such that $cg \in S\langle Y\rangle$. Corollary 10.12 tells us that $S\langle Y\rangle$ is generated as a ring over its subring $R\langle Y\rangle$ by S, so $cg = \sum_{j\in J} a_j g_j$ with finite J, and $a_j \in S$, $g_j \in R\langle Y\rangle$ for all $j \in J$, so $g = \sum_j c^{-1}a_j g_j$. Each $c^{-1}a_j$ is a K-linear combination of $b_1, \ldots, b_m$, so $g = b_1 f_1 + \cdots + b_m f_m$ with $f_1, \ldots, f_m \in K\langle Y\rangle$. Moreover, if $f_1, \ldots, f_m \in K\langle Y\rangle$ are not all zero, then $b_1 f_1 + \cdots + b_m f_m \neq 0$, by considering a monomial Y^ν for which one of the f_i has a nonzero coefficient. $\square$

10.3 Valuation A-rings

In this section A is noetherian with an ideal $\sigma(A) \neq A$, such that $\bigcap_e \sigma(A)^e = \{0\}$ and A is $\sigma(A)$-adically complete. Also, R is a *valuation A-ring*, that is, an A-ring whose underlying ring is a valuation ring. Thus $\sigma(A)R = \rho R$ for some $\rho \in \iota_0\big(\sigma(A)\big)$. We let $\sigma(R)$ denote the maximal ideal of R and $\boldsymbol{k} = R/\sigma(R)$ its residue field. Let K be the fraction field of R, viewed as a valued field given by its valuation ring R. We let $\mathcal{L}^A_{\preccurlyeq}$ be the language $\mathcal{L}^A$ of A-rings augmented by a binary relation symbol $\preccurlyeq$. We construe K as an $\mathcal{L}^A_{\preccurlyeq}$-structure by interpreting the symbols of the sublanguage $\{0, 1, -, +, \cdot\}$ in the usual way, interpreting the binary relation symbol $\preccurlyeq$ as described at the end of the Introduction for any valued field, and interpreting any n-ary function symbol $g \in A\langle Y_1, \dots, Y_n \rangle$ as the function

$$y \mapsto g(y) \text{ for } y \in R^n, \qquad y \mapsto 0 \text{ for } y \in K^n \setminus R^n.$$

Then R with its A-analytic structure and dominance relation $\preccurlyeq$ restricted to R is a substructure of the $\mathcal{L}^A_{\preccurlyeq}$-structure K. We define an *A-extension of K* to be a valued field extension of K whose valuation ring is equipped with an A-analytic structure that makes it an extension of the A-ring R. Thus any A-extension L of K is naturally an $\mathcal{L}^A_{\preccurlyeq}$-structure so that K is a substructure of L.

Viability. We define R to be *viable* if for some $t \in R^{\neq}$ we have

$$\mathcal{O}(R) \;=\; tR \;=\; \sqrt{\mathcal{O}(A)R}.$$

If R is viable and t is as above, then Γ has a least positive element, namely $v(t)$, and so R is not a field. In order to make our Weierstrass preparation and division theorems useful for the model theory of R as a valuation A-ring we need viability:

In the rest of this section we assume R is viable.

This assumption is satisfied if for some $t \in A$ we have $\mathcal{O}(A) = tA$ and $\mathcal{O}(R) = tR \neq \{0\}$. Our original interest was confined to this special case, but this stronger assumption is in general not inherited by the valuation ring of an A-extension of finite degree over K. The need to pass to such extensions in a key argument, namely the proof of Proposition 12.4, motivated this more general setting.

Note that R is henselian, by Lemma 10.1, so for any field extension F of K which is algebraic over K there is a unique valuation ring of F lying over R, and this valuation ring is the integral closure of R in F. Thus by Corollary 10.7:

Corollary 10.25. *If L is a valued field extension of K and is algebraic over K, then L has a unique expansion to an A-extension of K.*

In this corollary L might be an algebraic closure of K, in which case its valuation ring is the integral closure of R in L, and unlike the maximal ideal of R, the maximal ideal of this integral closure is not principal.

By the viability assumption on R our intended model theoretic results do not apply to algebraically closed valued fields whose valuation ring is equipped with an A-analytic structure. To avoid this assumption one could replace the restricted power series rings over A with rings of mixed power series over A where some variables range as before over the valuation ring and the other (formal) variables only over its maximal ideal. This is the direction taken by Lipshitz [44]; see also Lipshitz and Robinson [45]. Our treatment can probably be extended in this direction as well, but this will not be done here. We use the simpler device of passing from A to an extension $\widetilde{A}$ (introduced below) as a partial substitute.

We now fix $t \in R$ such that $v(t)$ is the least positive element of Γ, equivalently, $\mathcal{O}(R) = tR$. We identify $\mathbb{Z}$ with its image in Γ via $k \mapsto kv(t)$, so $v(t) = 1$ and $\mathbb{Z}$ is a convex subgroup of Γ.

Lemma 10.26. *Suppose L is an A-extension of K with $[L : K] < \infty$. Then the valuation A-ring R_L of L is viable.*

Proof. Since $[\Gamma_L : \Gamma] < \infty$, there are only finitely many positive elements of Γ_L less than vt. Thus Γ_L has a least positive element vs with $s \in L$. Then $s^n = tu$ with $n \in \mathbb{N}^{\geqslant 1}$ and $u \in R_L^\times$, so $s \in \sqrt{tR_L}$ and thus $\mathcal{o}(R_L) = sR_L = \sqrt{\mathcal{o}(A)R_L}$. ☐

Lemma 10.27. *Suppose K_0 is a valued subfield of K and its valuation ring $R_0 = R \cap K_0$ is an A-subring of R. Then the valuation A-ring R_0 is viable.*

Proof. Take $\rho \in \mathcal{o}(A)$ such that $\rho R = \mathcal{o}(A)R$. This gives $e \in \mathbb{N}^{\geqslant 1}$ with $t^e \preccurlyeq \iota_0(\rho)$, so $0 < v\big(\iota_0(\rho)\big) \leqslant e$. Hence $\Gamma_0 := v(K_0^\times) \subseteq \Gamma$ has a least positive element $\leqslant e$, and $\mathcal{o}(R_0)^e \subseteq \rho R_0$, and thus $\mathcal{o}(R_0) = \sqrt{\mathcal{o}(A)R_0}$. ☐

An A-extension of K is said to be *viable* if its valuation A-ring is viable.

Corollary 10.28. *Let L be a viable A-extension of K and let $z \in L$ be such that $t \prec z \prec 1$. Then $t^d \asymp z^e$ for some $e > d \geqslant 1$.*

Proof. Let γ be the least positive element of Γ_L, so $0 < \gamma \leqslant vz < 1 = v(t)$. Let K, L play the role of K_0, K in Lemma 10.27. The proof of that lemma then gives $e \geqslant 1$ such that $\gamma \leqslant 1 \leqslant e\gamma$, and so by decreasing e if necessary we arrange $e\gamma = 1$. Hence $vz = d\gamma$ with $1 \leqslant d < e$, and thus $t^d \asymp z^e$. ☐

It will be crucial to extend A to a ring $\widetilde{A}$ as follows: fix $t_1, \dots, t_r \in A$ such that $\mathcal{o}(A) = (t_1, \dots, t_r)$. Next, fix $\omega_1, \dots, \omega_r \in R$ and $e \in \mathbb{N}^{\geqslant 1}$ such that

$$t^e \;=\; t_1\omega_1 + \cdots + t_r\omega_r.$$

Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_r)$ be a tuple of r distinct indeterminates, also distinct from any indeterminates that we indicate below by roman capitals. The noetherian ring $A\langle\boldsymbol{\omega}\rangle = A\langle\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_r\rangle$ inherits the conditions imposed on A earlier, with $\mathcal{o}(A\langle\boldsymbol{\omega}\rangle) = (t_1, \dots, t_r)A\langle\boldsymbol{\omega}\rangle$, and we expand the A-ring R to an $A\langle\boldsymbol{\omega}\rangle$-ring by $f(\boldsymbol{\omega}, y) := f(\omega_1, \dots, \omega_r, y)$ for $f \in A\langle\boldsymbol{\omega}\rangle\langle Y\rangle = A\langle\boldsymbol{\omega}, Y\rangle$, in other words, we construe R as an (A, ω)-ring with $\omega = (\omega_1, \dots, \omega_r)$.

Next we set $p := T^e - (t_1\boldsymbol{\omega}_1 + \cdots + t_r\boldsymbol{\omega}_r) \in A\langle\boldsymbol{\omega}\rangle[T]$. Then $p(t) = 0$, so we can apply the subsection *Passing to $A\langle t_p\rangle$ and $A[[t_p]]$* of Section 10.1 to $A\langle\boldsymbol{\omega}\rangle$ in the role of A to make the $A\langle\boldsymbol{\omega}\rangle$-ring R into an $\widetilde{A}$-ring with $\widetilde{A} := A\langle\boldsymbol{\omega}\rangle[[t_p]]$: for $f_0, \dots, f_{e-1} \in A\langle\boldsymbol{\omega}\rangle\langle Y\rangle$ and $y \in R^n$,

$$\big(f_0 + t_p f_1 + \cdots + t_p^{e-1} f_{e-1}\big)(y) \;=\; f_0(y) + t f_1(y) + \cdots + t^{e-1} f_{e-1}(y).$$

We also remind the reader of Lemma 10.22, and recall an important consequence of Section 10.2: whether we view R as an A-ring or as an $\widetilde{A}$-ring makes no difference for what $R\langle Z\rangle$ is and for what $g(z) \in R^*$ is, where $g \in R\langle Z\rangle$, $z \in (R^*)^N$, and R^* is an A-ring extending R, with R^* construed in the obvious way as an $\widetilde{A}$-ring extending the $\widetilde{A}$-ring R. Note: R as an $\widetilde{A}$-ring is viable.

Weierstrass preparation and division with parameters. Let

$$f \;=\; \sum_\nu a_\nu(X) Y^\nu \in A\langle X, Y\rangle, \qquad n \geqslant 1.$$

We now study how Weierstrass preparation applies to $f(x, Y)$ for $x \in R^m$, and how this depends on x. Lemma 10.16 with $A\langle X\rangle$ in the role of A gives $d \geqslant 1$ and $b_{\mu\nu} \in \mathcal{o}(A\langle X\rangle)$ for $|\mu| < d$ and $|\nu| \geqslant d$. As before we set for $|\mu| < d$,

$$f_\mu \ := \ Y^\mu + \sum_{|\nu|\geqslant d} b_{\mu\nu} Y^\nu \in A\langle X, Y\rangle, \qquad f \ = \ \sum_{|\mu|<d} a_\mu f_\mu.$$

We order $\mathbb{N}^n$ lexicographically and for μ with $|\mu| < d$ we set

$$\begin{aligned} & I(\mu) \ := \ \{\lambda : \ |\lambda| < d, \ \lambda < \mu\}, \quad J(\mu) \ := \ \{\lambda : \ |\lambda| < d, \ \lambda > \mu\}, \ \text{ so} \\ (*) \qquad & f \ = \ \sum_{\lambda \in I(\mu)} a_\lambda f_\lambda + a_\mu f_\mu + \sum_{\lambda \in J(\mu)} a_\lambda f_\lambda. \end{aligned}$$

Now fix μ with $|\mu| < d$ and introduce tuples

$$U_\mu := \big(U_{\lambda\mu} : \lambda \in I(\mu)\big), \qquad V_\mu := \big(V_{\lambda\mu} : \lambda \in J(\mu)\big)$$

of indeterminates, different from each other and from the X_i and Y_j. Set

$$\begin{aligned} \widetilde{F}_\mu \ &:= \ \sum_{\lambda \in I(\mu)} U_{\lambda\mu} f_\lambda + f_\mu + \sum_{\lambda \in J(\mu)} t_p V_{\lambda\mu} f_\lambda \in \widetilde{A}\langle U_\mu, V_\mu, X, Y\rangle, \\ F_\mu \ &:= \ \widetilde{F}_\mu\big(U_\mu, V_\mu, X, T_d(Y)\big) \in \widetilde{A}\langle U_\mu, V_\mu, X, Y\rangle. \end{aligned}$$

Note that for $n = 1$ we have $T_d(Y) = Y$, so $F_\mu = \widetilde{F}_\mu$.

Lemma 10.29. *F_μ is regular of degree $\ell := \mu_1 d^{n-1} + \cdots + \mu_n$ in Y_n, and so*

$$F_\mu \ = \ E \cdot (Y_n^\ell + G_1 Y_n^{\ell-1} + \cdots + G_\ell)$$

for a unit E of $\widetilde{A}\langle U_\mu, V_\mu, X, Y\rangle$ and suitable $G_1, \ldots, G_\ell \in \widetilde{A}\langle U_\mu, V_\mu, X, Y'\rangle$.

Here is a consequence of Lemma 10.29 for $n = 1$ (so $Y = Y_1$):

Corollary 10.30. *Let $n = 1$ and $g(Y) = \sum_{j=0}^\infty c_j Y^j \in K\langle Y\rangle$, $g \neq 0$. Then:*

- *(i) there is $\mu \in \mathbb{N}$ with $c_i \preccurlyeq c_\mu \succ c_j$ whenever $i \leqslant \mu < j$;*
- *(ii) for the unique μ in* (i) *we have $g(Y) = c \cdot r(Y) \cdot (Y^\mu + g_1 Y^{\mu-1} + \cdots + g_\mu)$ with $c = c_\mu \in K^\times$, $r(Y) \in R\langle Y\rangle^\times$, and $g_1, \ldots, g_\mu \in R$.*

Proof. We multiply g by an element of $K^\times$ to arrange $g \in R\langle Y\rangle$. Then $g(Y) = f(x, Y)$ with $x \in R^m$ and $f = f(X, Y) = \sum_j a_j(X) Y^j$ in $A\langle X, Y\rangle$, so $c_j = a_j(x)$ for all j. Lemma 10.16 with $A\langle X\rangle$ in the role of A gives $d \geqslant 1$ and $b_{ij} \in \mathcal{O}(A\langle X\rangle)$ for $i < d \leqslant j$ such that $a_j = \sum_{i<d} a_i b_{ij}$ for all $j \geqslant d$. Set

$$\gamma \ := \ \min_{i<d} v(c_i), \qquad \mu \ := \ \max\{i < d : \ v(c_i) = \gamma\}.$$

Then (i) holds for this μ: for $\mu < j$, distinguish the cases $j < d$ and $j \geqslant d$.

For (ii) we use the identities above for $n = 1$ and our f. The identity $(*)$ yields $f = \sum_{i<\mu} a_i f_i + a_\mu f_\mu + \sum_{\mu<i<d} a_i f_i$. Substituting x for X and factoring out $c := c_\mu = a_\mu(x)$ (possible because $c \neq 0$) gives

$$c^{-1} g(Y) \ = \ \sum_{i<\mu} (c_i/c) f_i(x, Y) + f_\mu(x, Y) + \sum_{\mu<i<d} (c_i/c) f_i(x, Y),$$

so for $u := \big(c_i/c : i < \mu\big) \in R^{\mu}$ and $v := \big(c_i/tc : \ \mu < i < d\big) \in R^{d-1-\mu}$ we have $c^{-1}g(Y) = F_{\mu}(u,v,x,Y)$. Now applying Lemma 10.29 for $n=1$ shows that (ii) holds with $r(Y) = E(u,v,x,Y)$ and $g_i = G_i(u,v,x)$ for $i = 1,\dots,\mu$. $\square$

Note that the proof above uses in a crucial way that $\mathcal{o}(R) = tR$.

Corollary 10.31. *Let R^* be an A-ring extending R, and suppose $y \in R^*$ is not integral over R. Then $R\langle y\rangle$ has the following properties, with $n=1$ in* (i)*:*

(i) the morphism $g(Y) \mapsto g(y) : R\langle Y\rangle \to R\langle y\rangle$ of A-rings is an isomorphism;

(ii) $R\langle y\rangle$ is a domain but not a valuation ring;

(iii) inside the ambient field $\operatorname{Frac}(R\langle y\rangle)$ we have $R\langle y\rangle \nsubseteq K(y)$.

Proof. For (ii), use that $Y \notin tR\langle Y\rangle$ and $t \notin YR\langle Y\rangle$. For (iii), if $\operatorname{char}\boldsymbol{k} \neq 2$, then the polynomial $Z^2 - (1+ty)$ has a zero in $R\langle y\rangle$ by Corollary 10.15, but has no zero in $K(y)$. If $\operatorname{char}\boldsymbol{k} = 2$, use instead the polynomial $Z^3 - (1+ty)$. $\square$

We return to our $f(X,Y) \in A\langle X,Y\rangle$ with $n \geqslant 1$. To find out how Weierstrass preparation for $f(x,Y)$ depends on $x \in R^m$, we now introduce the quantifier-free $\mathcal{L}^A_{\preccurlyeq}$-formulas $Z(X)$ and $S_{\mu}(X)$ (for $|\mu| < d$) in the variables X:

$$
\begin{aligned}
Z(X) &:= \bigwedge_{|\mu|<d} a_{\mu}(X) = 0,\\
S_{\mu}(X) &:= a_{\mu}(X) \neq 0 \wedge \big(\bigwedge_{\lambda \in I(\mu)} a_{\lambda}(X) \preccurlyeq a_{\mu}(X)\big) \wedge \big(\bigwedge_{\mu \in J(\mu)} a_{\lambda}(X) \prec a_{\mu}(X) \ \big).
\end{aligned}
$$

Lemma 10.32. *For the $\mathcal{L}^A_{\preccurlyeq}$-structure R we have the following:*

(i) for all $x \in R^m$, $Z(x)$ holds or $S_{\mu}(x)$ holds for some μ with $|\mu| < d$;

(ii) suppose $x \in R^m$, $|\mu| < d$, and $S_{\mu}(x)$ holds; so $u_{\lambda\mu} := a_{\lambda}(x)/a_{\mu}(x) \in R$ for $\lambda \in I(\mu)$ and $v_{\lambda\mu} := a_{\lambda}(x)/ta_{\mu}(x) \in R$ for $\lambda \in J(\mu)$. Then with

$$
u_{\mu} \ := \ \big(u_{\lambda\mu} : \lambda \in I(\mu)\big), \qquad v_{\mu} \ := \ \big(v_{\lambda\mu} : \lambda \in J(\mu)\big),
$$

and $E, G_1, \dots, G_{\ell}$ as in Lemma 10.29 we have

$$
f\big(x, T_d(Y)\big) \ = \ a_{\mu}(x) F_{\mu}(u_{\mu}, v_{\mu}, x, Y) \ \text{in } R\langle Y\rangle
$$

and $F_{\mu}(u_{\mu}, v_{\mu}, x, Y)$ equals, in $R\langle Y\rangle$, the product

$$
E(u_{\mu}, v_{\mu}, x, Y) \cdot \big(Y_n^{\ell} + G_1(u_{\mu}, v_{\mu}, x, Y')Y_n^{\ell-1} + \cdots + G_{\ell}(u_{\mu}, v_{\mu}, x, Y')\big).
$$

We can now prove a converse of Lemma 10.17:

Lemma 10.33. *Suppose $x \in R^m$ and $f(x,y) = 0$ for all $y \in R^n$. Then $f(x,Y) = 0$.*

Proof. If $Z(x)$ holds, then $a_\nu(x) = 0$ for all ν, that is, $f(x, Y) = 0$. Next assume $|\mu| < d$ and $S_\mu(x) \neq 0$. Then by (ii) of Lemma 10.32 we have a monic polynomial in $R[Y_n]$ vanishing identically on R. This is impossible as R is infinite. $\square$

Corollary 10.34. *If $g \in R\langle Y\rangle$ and $g(y) = 0$ for all $y \in R^n$, then $g = 0$.*

By the last corollary, the map

$$K\langle Y\rangle \to \text{ring of } K\text{-valued functions on } R^n$$

that assigns to each $g \in K\langle Y\rangle$ the function $y \mapsto g(y)$ is an injective morphism of K-algebras.

Consequences for $K\langle Y\rangle$ of Weierstrass division. For an algebraic closure K^{a} of K, the integral closure R^{a} of R in K^{a} is the unique valuation ring of K^{a} dominating R, and has a unique A-analytic structure extending that of R.

More generally, we fix below an algebraically closed valued field extension K^{a} of K (not necessarily an algebraic closure of K), whose valuation ring R^{a} is equipped with an A-analytic structure extending that of R. This gives rise to $K\langle Y\rangle \subseteq K^{\mathrm{a}}\langle Y\rangle$ and for $y \in (R^{\mathrm{a}})^n$ we have the evaluation map $g \mapsto g(y) : K^{\mathrm{a}}\langle Y\rangle \to K^{\mathrm{a}}$, which for $y \in R^n$ extends the previous evaluation map $K\langle Y\rangle \to K$.

Lemma 10.35. *If E is a unit of $R\langle Y\rangle$, then $E(y) \asymp 1$ for all $y \in (R^{\mathrm{a}})^n$.*

This is clear. The next two lemmas follow easily from $(*)$ and Lemma 10.32.

Lemma 10.36. *Let $g(Y) = \sum_\nu c_\nu Y^\nu \in R\langle Y\rangle$, $g \neq 0$. Then:*

(i) there is a $d \geqslant 1$ and an index $\mu \in \mathbb{N}^n$ with $|\mu| < d$ such that

$$c_\nu \preccurlyeq c_\mu \text{ whenever } |\nu| < d, \quad c_\nu \prec c_\mu \text{ whenever } |\nu| \geqslant d;$$

(ii) if $c_\nu \prec 1$ for all ν, then $g(y) \prec 1$ for all $y \in (R^{\mathrm{a}})^n$.

Lemma 10.37. *Let $g(Y) \in K\langle Y\rangle^{\neq}$, $n \geqslant 1$. Then for some $d \in \mathbb{N}^{\geqslant 1}$ and $\ell \in \mathbb{N}$,*

(i) $g\big(T_d(Y)\big) \ = \ c \cdot E(Y) \cdot \big(Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \cdots + c_l(Y')\big)$
where $c \in K^\times$, $E(Y) \in R\langle Y\rangle$ is a unit, and $c_1(Y'), \dots, c_\ell(Y') \in R\langle Y'\rangle$.

(ii) $R\langle Y\rangle \ = \ \big(Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \cdots + c_l(Y')\big)R\langle Y\rangle + \sum_{i<\ell} R\langle Y'\rangle Y_n^i$ and

$$K\langle Y\rangle \ = \ g\big(T_d(Y)\big)K\langle Y\rangle + \sum_{i<\ell} K\langle Y'\rangle Y_n^i.$$

Proof. For (ii), use a reduction to $A\langle X, Y\rangle$ and appeal to Lemma 9.15. $\square$

Here is a variant where we can dispense with a transformation T_d:

Proposition 10.38. *Let $g \in K\langle Y\rangle$, $n \geqslant 1$, $g = g_1 + \cdots + g_n$, $g_j = g_j(Y_j) \in K\langle Y_j\rangle$ for $j = 1, \dots, n$ and $g_j \neq 0$ for some j. Then for some $j \in \{1, \dots, n\}$ and $\ell \in \mathbb{N}$,*

$$g(Y) \ = \ c \cdot E(Y) \cdot \big(Y_j^\ell + c_1(Y^*)Y_j^{\ell-1} + \cdots + c_l(Y^*)\big)$$

where $c \in K^\times$, $E \in R\langle Y\rangle$ is a unit, $Y^* = (Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_n)$, and where $c_1(Y^*), \dots, c_l(Y^*) \in R\langle Y^*\rangle$.

Proof. Multiplying by a factor from $K^\times$ we arrange that $g_j \in R\langle Y_j\rangle$ for $j = 1,\dots,n$, and the image of g_j in $\boldsymbol{k}[Y_j]$ is nonzero for some j, say for $j = n$. Take m, $x \in R^m$, and $G_j(X,Y_j) \in A\langle X,Y_j\rangle$ such that $g_j(Y_j) = G_j(x,Y_j)$ for $j = 1,\dots,n$. Then $g = G(x,Y)$ for $G := G_1 + \cdots + G_n \in A\langle X,Y\rangle$. We now proceed to a simpler version of the construction in the beginning of the previous subsection.

We have $G_n = \sum_{l=0}^\infty a_l(X)Y_n^l$ with all $a_l \in A\langle X\rangle$, and so $g_n = \sum_{l=0}^\infty a_l(x)Y_n^l$. Take $\mu \in \mathbb{N}$ such that $a_\mu(x) \asymp 1$ and $a_l(x) \prec 1$ for all $l > \mu$. Take $d \in \mathbb{N}^{>\mu}$ and $b_{kl} \in \mathcal{o}(A\langle X\rangle)$ for $k < d \leqslant l$ such that $a_l = \sum_{k<d} a_k b_{kl}$ for $l \geqslant d$ and $b_{kl} \to 0$ as $l \to \infty$ for each fixed $k < d$. For $i < d$ we set $f_i := Y_n^i + \sum_{l\geqslant d} b_{il}Y_n^l \in A\langle X,Y_n\rangle$, so

$$G_n \;=\; \sum_{i<\mu} a_i f_i + a_\mu f_\mu + \sum_{\mu<i<d} a_i f_i.$$

Let $U_0,\dots,U_{\mu-1},V_{\mu+1},\dots,V_{d-1},W$ be distinct indeterminates, also different from $X_1,X_2,\dots,Y_1,Y_2,\dots$, set $U := (U_i : i < \mu)$, $V = (V_i : \mu < i < d)$, and

$$\widetilde{G} \;:=\; W\cdot(G_1 + \cdots + G_{n-1}) + \sum_{i<\mu} U_i f_i + f_\mu + \sum_{\mu<i<d} t_p V_i f_i \in \widetilde{A}\langle U,V,W,X,Y\rangle.$$

Then for $u := \big(a_i(x)/a_\mu(x)\big)_{i<\mu}$, $v := \big(a_i(x)/ta_\mu(x)\big)_{\mu<i<d}$, $w := 1/a_\mu(x)$, we have

$$g \;=\; G(x,Y) \;=\; a_\mu(x)\cdot \widetilde{G}(u,v,w,x,Y).$$

It remains to note that $\widetilde{G}$ is regular in Y_n of degree μ, and to use Corollary 9.19. $\square$

Weierstrass division leads in the usual way to noetherianity of $K\langle Y\rangle$ and more:

Theorem 10.39. *The integral domain $K\langle Y\rangle$ has the following properties:*

(i) $K\langle Y\rangle$ is noetherian, and for every proper ideal I of $K\langle Y\rangle$:

(ii) there is an injective K-algebra morphism $K\langle Y_1,\dots,Y_m\rangle \to K\langle Y\rangle/I$ with $m \leqslant n$, making $K\langle Y\rangle/I$ into a finitely generated $K\langle Y_1,\dots,Y_m\rangle$-module;

(iii) there is $y \in (R^{\mathrm{a}})^n$ such that $f(y) = 0$ for all $f \in I$.

Proof. By induction on n. The case $n = 0$ being obvious, let $n \geqslant 1$. Recall that for $d \in \mathbb{N}^{\geqslant 1}$ we have the automorphism $g(Y) \mapsto g\big(T_d(Y)\big)$ of the K-algebra $K\langle Y\rangle$. Let I be an ideal of $K\langle Y\rangle$, $I \neq \{0\}$. Take a nonzero $g \in I$. To show I is finitely generated we apply an automorphism as above and use Lemma 10.37 to arrange $g = Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \cdots + c_\ell(Y')$ with $\ell \in \mathbb{N}$, $c_1,\dots,c_\ell \in R\langle Y'\rangle$, and

$$R\langle Y\rangle \;=\; gR\langle Y\rangle + \sum_{i<\ell} R\langle Y'\rangle Y_n^i, \qquad K\langle Y\rangle \;=\; gK\langle Y\rangle + \sum_{i<\ell} K\langle Y'\rangle Y_n^i.$$

For $\ell = 0$ this means $g = 1$, and we are done, so assume $\ell \geqslant 1$. Then the inclusion $K\langle Y'\rangle \to K\langle Y\rangle$ followed by the canonical map $K\langle Y\rangle \to K\langle Y\rangle/(g)$ makes $K\langle Y\rangle/(g)$ a $K\langle Y'\rangle$-module that is generated by the images of the Y_n^i with $i < \ell$. Assuming inductively that $K\langle Y'\rangle$ is noetherian, it follows that $K\langle Y\rangle/(g)$ is noetherian as a $K\langle Y'\rangle$-module, and thus as a ring. Hence the image of I in $K\langle Y\rangle/(g)$ is finitely generated, say by the images of $g_1,\dots,g_k \in I$, $k \in \mathbb{N}$. Then I is generated by $g, g_1,\dots,g_k$. This proves noetherianity of $K\langle Y\rangle$. Let now I also be proper, that is, $1 \notin I$, and set $I' := I \cap K\langle Y'\rangle$. The natural K-algebra embedding $K\langle Y'\rangle/I' \to K\langle Y\rangle/I$ makes $K\langle Y\rangle/I$ a finitely generated $K\langle Y'\rangle/I'$-module by the above. Assuming inductively that (ii) holds for

$n-1,\ K\langle Y'\rangle,\ I'$ instead of $n, K\langle Y\rangle, I$ yields (ii). For (iii) we can arrange that I is a maximal ideal of $K\langle Y\rangle$. Then in (ii) we have $m = 0$, so $K\langle Y\rangle/I$ is finite-dimensional as a vector space over K, hence algebraic over K as a field extension of K. This gives a K-algebra morphism $\phi : K\langle Y\rangle \to K^{\mathrm{a}}$ with kernel I and $\phi(K\langle Y\rangle)$ algebraic over K. We set $y := (y_1, \dots, y_n) = \big(\phi(Y_1), \dots, \phi(Y_n)\big) \in (K^{\mathrm{a}})^n$. We claim that $\phi(R\langle Y\rangle) \subseteq R^{\mathrm{a}}$ (and thus $\phi(R\langle Y\rangle)$ is integral over R).

Using $\phi(g) = 0$ gives $\phi(R\langle Y\rangle) = \sum_{i<\ell} \phi(R\langle Y'\rangle) y_n^i$. Since I' is a maximal ideal of $K\langle Y'\rangle$ we can assume inductively that $\phi(R\langle Y'\rangle) \subseteq R^{\mathrm{a}}$, so $\phi(R\langle Y\rangle) \subseteq \sum_{i<\ell} R^{\mathrm{a}} y_n^i$. Now $\phi(g) = 0$ means

$$y_n^\ell + \phi\big(c_1(Y')\big) y_n^{\ell-1} + \cdots + \phi\big(c_\ell(Y')\big) \ = \ 0,$$

with $\phi\big(c_j(Y')\big) \in R^{\mathrm{a}}$ for $j = 1, \dots, \ell$. Hence $y_n \in R^{\mathrm{a}}$, which proves the claim. Therefore $y \in (R^{\mathrm{a}})^n$, and by Corollary 10.8 the restriction of ϕ to a map $R\langle Y\rangle \to R^{\mathrm{a}}$ is a morphism of A-rings. Thus for $f(Y) \in R\langle Y\rangle$ we have $\phi\big(f(Y)\big) = f(y)$, in particular, $f(y) = 0$ for all $f \in I$. $\square$

10.4 Immediate A-extensions

The study of immediate extensions of valued fields plays a key role in proving AKE-results via model theory and valuation theory. We try to follow this pattern. By Lemma 10.1 and Corollary 10.25, the case of algebraic immediate extensions is under control (at least in the equicharacteristic 0 case), so we are left with proving that a pc-sequence of transcendental type "generates" an immediate extension. The problem is that the valuation ring of such an extension should now be an A-ring, and thus closed under many more operations than in the usual setting. In this section we show how to overcome this problem. This section uses only the material of Section 10.3 that precedes Lemma 10.32.

Below we assume some familiarity with [29, Section 4]; when using a result from those lecture notes we shall indicate the specific reference.

We continue with the previously set assumptions on A and R: *A is noetherian with an ideal $\smallo(A) \neq A$ such that $\bigcap_e \smallo(A)^e = \{0\}$ and A is $\smallo(A)$-adically complete; R is a viable valuation A-ring.*

We fix $t \in R$ such that $\smallo(R) = tR$ and adopt the notations and terminology concerning R and its fraction field K from Section 10.3, with the valuation $v : K^\times \to \Gamma$ on K such that $R = \{a \in K :\ va \geqslant 0\}$, so vt is the least positive element of Γ. For any valued field extension L of K we let $\Gamma_L \supseteq \Gamma$ be the value group of L and denote the valuation of L also by v, so that $v : L^\times \to \Gamma_L$ extends $v : K^\times \to \Gamma$.

By [29, Lemma 4.3] and the remark following its proof, any pc-sequence in K has a pseudolimit in some elementary $\mathcal{L}^A_{\preccurlyeq}$-extension of K; any such extension is an A-extension of K whose valuation ring inherits the conditions we imposed on R.

Immediate A-extensions generated by a pseudocauchy sequence. In this subsection L is an A-extension of K. Thus the valuation A-ring S of L extends the A-ring R and dominates R. We also view any subfield F of L as a *valued* subfield of L, and thus as a valued field extension of K if $K \subseteq F$.

Let (a_ρ) be a pc-sequence in K of transcendental type over K, with all $a_\rho \in R$, and with pseudolimit $a \in L$. Then $a \in S$, a is transcendental over K, and the valued subfield $K(a)$ of L is an immediate extension of K, by [29, Theorem 4.9]. But the valuation ring of $K(a)$ does not contain $R\langle a\rangle$ by Corollary 10.31, and so is not A-closed in S.

Is there a valued subfield $K_a \supseteq K \cup \{a\}$ of L that is an immediate extension of K and whose valuation ring R_a is A-closed in S? Such R_a must contain $R\langle a\rangle$, but has to be strictly larger, since $R\langle a\rangle$ is not a valuation ring, by Corollary 10.31.

To answer the question above affirmatively we proceed as follows. Take an index ρ_0 such that for $\rho > \rho_0$,

$$a \;=\; a_\rho + t_\rho u_\rho, \quad t_\rho \in K^\times,\; t_\rho \prec 1,\; u_\rho \in K(a),\; u_\rho \asymp 1,$$

and $v(t_\rho)$ is strictly increasing as a function of $\rho > \rho_0$. Then for indices $\sigma > \rho > \rho_0$ we have $R[u_\rho] \subseteq R[u_\sigma]$, and thus

$$R\langle a\rangle \;\subseteq\; R\langle u_\rho\rangle \;\subseteq\; R\langle u_\sigma\rangle.$$

This yields an A-closed subring $R_a := \bigcup_{\rho>\rho_0} R\langle u_\rho\rangle$ of S. Note that R_a does not change upon increasing ρ_0, and the next proposition shows more: as the notation suggests, R_a depends only on R and a, not on (a_ρ).

Proposition 10.40. *The subring R_a of S has the following properties:*

(i) the valued subfield $K_a := \mathrm{Frac}(R_a)$ of L is an immediate extension of K;

(ii) R_a is the least A-closed subring of S, with respect to inclusion, that contains $R \cup \{a\}$ and is a valuation ring dominated by S;

Proof. Let $P \in K[Y] \setminus K$ where $n = 1$, so $Y = Y_1$. Let I be the set of i in $\{1, \dots, \deg P\}$ with $P_{(i)}(Y) \neq 0$. Then $I \neq \emptyset$ and for all $\rho > \rho_0$,

$$P(a) \;=\; P(a_\rho) + \sum_{i\in I} P_{(i)}(a_\rho)(a - a_\rho)^i \;=\; P(a_\rho) + \sum_{i\in I} t_\rho^i P_{(i)}(a_\rho) u_\rho^i.$$

The proof of [29, Proposition 4.7] gives $i_0 \in I$ such that, eventually,

$$\begin{aligned} \text{for all } i \in I \setminus \{i_0\}, \quad t_\rho^{i_0} P_{(i_0)}(a_\rho) \;&\succ\; t_\rho^i P_{(i)}(a_\rho), \\ P(a) - P(a_\rho) \;&\sim\; t_\rho^{i_0} P_{(i_0)}(a_\rho), \end{aligned}$$

and $v\big(t_\rho^{i_0} P_{(i_0)}(a_\rho)\big) = v\big(P(a) - P(a_\rho)\big)$ is eventually strictly increasing. Now (a_ρ) is of transcendental type over K, so $v\big(P(a_\rho)\big)$ is eventually constant, and thus $P(a_\rho) \succ P(a) - P(a_\rho)$, eventually. Thus eventually,

$$P(a) \;=\; P(a_\rho) \cdot \big(1 + \sum_{i\in I} \frac{t_\rho^i P_{(i)}(a_\rho)}{P(a_\rho)} u_\rho^i\big) \;\in\; P(a_\rho) \cdot (1 + tR\langle u_\rho\rangle).$$

Now suppose $Q(Y) \in K[Y]^{\neq}$. Then likewise we have for $j = 1, \dots, \deg Q$ that eventually $0 \neq Q(a_\rho) \succ t_\rho^j Q_{(j)}(a_\rho)$, so eventually

$$Q(a) \;=\; Q(a_\rho) \cdot \big(1 + \sum_{j=1}^{\deg Q} \frac{t_\rho^j Q_{(j)}(a_\rho)}{Q(a_\rho)} u_\rho^j\big) \;\in\; Q(a_\rho) \cdot (1 + tR\langle u_\rho\rangle).$$

Therefore, if $P(a) \preccurlyeq Q(a)$, then eventually $\frac{P(a_\rho)}{Q(a_\rho)} \in R$, and so eventually

$$\frac{P(a)}{Q(a)} \;\in\; \frac{P(a_\rho)}{Q(a_\rho)} \cdot (1 + tR\langle u_\rho\rangle) \;\subseteq\; R \cdot (1 + tR\langle u_\rho\rangle) \;\subseteq\; R\langle u_\rho\rangle.$$

Thus the valuation ring of the valued subfield $K(a)$ of L is contained in R_a. Now we use the reduction to polynomials from Corollary 10.30(ii) to the effect that for g,h in $R\langle Y\rangle$ with $h\neq 0$, if $g(a)\preccurlyeq h(a)$, then $g(a)/h(a)\in R_a$. Thus the valuation ring of the valued subfield $\operatorname{Frac}(R\langle a\rangle)$ of L is contained in R_a, and it also follows from the last display that $\operatorname{Frac}(R\langle a\rangle)$ is an immediate extension of K.

Next, fix $\rho>\rho_0$ and note that for $\sigma>\rho$ we have

$$u_\rho \;=\; a_{\sigma\rho}+t_{\sigma\rho}u_\sigma, \qquad a_{\sigma\rho} \;:=\; \frac{a_\sigma-a_\rho}{t_\rho}\in R, \qquad t_{\sigma\rho} \;:=\; \frac{t_\sigma}{t_\rho},$$

and $(a_{\sigma\rho})_{\sigma>\rho}$ is a pc-sequence in K and of transcendental type over K such that $a_{\sigma\rho}\rightsquigarrow \frac{a-a_\rho}{t_\rho}=u_\rho$. Hence the above arguments applied to u_ρ instead of a show that the valuation ring of $\operatorname{Frac}(R\langle u_\rho\rangle)$ as a valued subfield of L is contained in $\bigcup_{\sigma>\rho}R\langle u_\sigma\rangle=R_a$, and is an immediate extension of K. Taking the union over all $\rho>\rho_0$ and using $R_a\subseteq S$ yields that R_a is the valuation ring of the valued subfield $K_a:=\operatorname{Frac}(R_a)$ of L, and that K_a is an immediate extension of K. This proves (i) and also shows that S dominates R_a.

As to (ii), let R^* be any A-closed subring of S containing $R\cup\{a\}$ such that R^* is a a valuation ring dominated by S. Then clearly $u_\rho\in R^*$ for all $\rho>\rho_0$, and thus $R_a\subseteq R^*$. $\square$

We keep (a_ρ) for now, and show that K_a is essentially unique:

Corollary 10.41. *Let L' be an A-extension of K with valuation A-ring S'. Suppose $a_\rho\rightsquigarrow a'\in S'$, thus giving rise to $R_{a'}\subseteq K_{a'}\subseteq L'$. Then there is a unique isomorphism $R_a\to R_{a'}$ of A-rings that is the identity on R and sends a to a'. It extends to a valued field isomorphism $K_a\to K_{a'}$.*

Proof. Using notations from the proof of Proposition 10.40 we have $a'=a_\rho+t_\rho u'_\rho$ with $u'_\rho\in K(a')$, $u'_\rho\asymp 1$ for $\rho>\rho_0$. That same proof and Corollary 10.31 yields for all $\rho>\rho_0$ a unique isomorphism $R\langle u_\rho\rangle\to R\langle u'_\rho\rangle$ of A-rings that is the identity on R and sends u_ρ to u'_ρ. Moreover, for $\sigma>\rho>\rho_0$ we have

$$u_\rho \;=\; a_{\sigma\rho}+t_{\sigma\rho}u_\sigma, \qquad u'_\rho \;=\; a_{\sigma\rho}+t_{\sigma\rho}u'_\sigma,$$

and so the above isomorphism $R\langle u_\sigma\rangle\to R\langle u'_\sigma\rangle$ extends the above isomorphism $R\langle u_\rho\rangle\to R\langle u'_\rho\rangle$. Taking the union over all $\rho>\rho_0$ yields an isomorphism $R_a\to R_{a'}$ of A-rings that is the identity on R and sends a to a'. Any such isomorphism sends u_ρ to u'_ρ for $\rho>\rho_0$, and this gives uniqueness. Now R_a and $R_{a'}$ are the valuation rings of K_a and $K_{a'}$, so this isomorphism $R_a\to R_{a'}$ extends to an isomorphism $K_a\to K_{a'}$ of valued fields. $\square$

Uniqueness of maximal immediate extensions over A. The results in this subsection about maximal immediate A-extensions will not be used later, but are included for their intrinsic interest. So far we did not restrict the characteristic of $\boldsymbol{k}$ or K, but now we also assume:

Either $\operatorname{char}(\boldsymbol{k})=0$ (*the equicharacteristic* 0 *case*), *or* K *as a valued field is finitely ramified of mixed characteristic.*

This is a well-known sufficient condition for an ordinary valued field to have an essentially unique maximal immediate extension; see [29, 4.29]. We now adapt this to our A-setting. A first consequence of the present assumptions is that K has no proper algebraic immediate A-extension, by [29, Corollary 4.22]. Note that any immediate A-extension of K inherits all the conditions we imposed so far on K. By a *maximal immediate A-extension of* K we mean an immediate A-extension L of K such that L has no proper immediate A-extension.

The previous subsection, the nonexistence of proper algebraic immediate A-extensions of K, and [29, Section 4] yield for an immediate A-extension L of K that the following are equivalent:

1. L is a maximal immediate A-extension of K,
2. L is maximal as a valued field,
3. L is spherically complete.

Corollary 10.42. *K has a maximal immediate A-extension, and such an extension is unique up to $\mathcal{L}^A_{\preccurlyeq}$-isomorphism over K.*

Proof. This goes along the same lines as the proof for ordinary valued fields: First, existence of a maximal immediate A-extension of K follows by Zorn and Krull's cardinality bound, like [29, Corollary 4.14]. As to uniqueness, using Corollary 10.41 this goes as in the proof of [29, Corollary 4.29]. □

Using Corollary 10.41 we obtain in the same way:

Corollary 10.43. *Any maximal immediate A-extension of K can be embedded, as an $\mathcal{L}^A_{\preccurlyeq}$-structure, into any $|\Gamma|^+$-saturated A-extension of K.*

CHAPTER 11

A theory of affinoids

11.1 Affinoids

Here we define the affinoid subsets of the projective line over an algebraically closed valued field with a nontrivial valuation, using suggestive multiplicative notation (as if we were dealing with an absolute value). Next we show that an inequality $|r(z)| \leqslant 1$ given by a rational function $r(Z)$ in one variable Z over such a valued field defines such an affinoid. In the presence of a suitable analytic A-structure on the valuation ring of a valued subfield we introduce in the next section for certain affinoids F the so-called affinoid algebra of functions on F. These functions on F will turn out to be closely related to rational functions.

Our treatment of these notions here is self-contained, though we borrow much in this and the next section from [37] where the valuation is assumed to be complete of rank 1. We avoid this assumption here, since it is not of a first-order nature in the logical sense and we need the results later in a first-order setting.

The projective line. We begin with some generalities not involving the field being algebraically closed or equipped with a valuation. Let K be a field. We define the projective line $\mathbb{P} = \mathbb{P}(K)$ to be the set of one-dimensional linear subspaces of the K-linear space K^2, so its points are the lines $[x_0 : x_1] := K(x_0, x_1)$ with $(x_0, x_1) \in K^2 \setminus \{(0,0)\}$. We identify $\lambda \in K$ with the point $[1 : \lambda]$ of $\mathbb{P}$, so $\mathbb{P} = K \cup \{\infty\}$ where $\infty := [0 : 1]$. We make the group $\mathrm{GL}_2(K)$ act on $\mathbb{P}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x_0 : x_1] \ := \ [cx_1 + dx_0 : ax_1 + bx_0],$$

or perhaps more readably: for $z \in \mathbb{P} = K \cup \{\infty\}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \ := \ \frac{az + b}{cz + d}.$$

with the usual conventions: for $z = \infty$, the right hand side takes the value $\frac{a}{c}$ ($= \infty$ for $c = 0$), and for $z \in K$ it takes the value ∞ iff $cz + d = 0$). Instead of $\frac{aZ+b}{cZ+d}$ we may consider any rational function $r(Z) \in K(Z)$. Then r defines a map

$$z \mapsto r(z) \ : \ \mathbb{P} \to \mathbb{P},$$

with the more general convention that for $r = p/q$ and coprime $p, q \in K[Z]$ with monic q we set $r(\infty) :=$ leading coefficient of p for $\deg p = \deg q$, $r(\infty) = 0$ for $\deg p < \deg q$, $r(\infty) = \infty$ for $\deg p > \deg q$, and for $z \in K$: $r(z) = \infty$ iff $q(z) = 0$, and $r(z) = p(z)/q(z)$ if $q(z) \neq 0$.) (Recall in this connection that for $z \in \mathbb{P}$ we have a unique discrete valuation

$$v_z \ : \ K(Z)^\times \to \mathbb{Z},$$

trivial on K, such that $v_z(Z-z)=1$ if $z \in K$, and $v_\infty(Z^{-1})=1$, and that the above convention amounts to setting $r(z)=\infty$ iff $v_z(r(Z))<0$; identifying K in the usual way with the residue field of v_z we note that if $v_z(r(Z)) \geqslant 0$, then $r(z) \in K$ is the image of $r(Z)$ in this residue field.)

The upper triangular matrices in $\mathrm{GL}_2(K)$ make up a subgroup of $\mathrm{GL}_2(K)$, the *affine subgroup of* $\mathrm{GL}_2(K)$. This subgroup consist exactly of the $g \in \mathrm{GL}_2(K)$ that fix the point $\infty \in \mathbb{P}$ (with respect to the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}$) and the resulting action of this subgroup on K is doubly transitive: for all $x_1 \neq x_2$ and $y_1 \neq y_2$ in K there is a g in this subgroup such that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$. It is also easy to check that for any $x \in \mathbb{P} \setminus \{0,1\}$ there is a $g \in \mathrm{GL}_2(K)$ with $g \cdot 0 = 0,\ g \cdot 1 = 1$, and $g \cdot \infty = x$. It follows easily that the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}$ is triply transitive: for any three-element subsets $\{x_1,x_2,x_3\},\{y_1,y_2,y_3\}$ of $\mathbb{P}$ there is $g \in \mathrm{GL}_2(K)$ such that $g \cdot x_i = y_i$ for $i=1,2,3$.

Under the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}$, the normal subgroup $K^\times I$ of $\mathrm{GL}_2(K)$ with I the identity of $\mathrm{GL}_2(K)$ acts trivially, inducing an action of $\mathrm{GL}_2(K)/K^\times I$ on $\mathbb{P}$.

The valued field setting. Here K is a field equipped with a valuation. We use multiplicative notation, so the valuation is a surjective function $z \mapsto |z| : K \to |K|$ (like an absolute value), where $|K| = |K^\times| \cup \{0\}$ is a totally ordered set with least element $0 \notin |K^\times|$ and $|K^\times|$ with the induced ordering is an ordered (multiplicative) commutative group, such that $|a|=0$ iff $a=0$, $|a+b| \leqslant \max(|a|,|b|)$, and $|ab|=|a|\cdot|b|$ for $a,b \in K$ (where by convention $0 \cdot \rho = \rho \cdot 0 := 0 \in |K|$ for $\rho \in |K|$.) We extend this valuation to $\mathbb{P}$ by adjoining to $|K|$ an element $\infty \notin |K|$, extending the total ordering of $|K|$ to $|K|_\infty = |K| \cup \{\infty\}$ so that $\rho < \infty$ for all $\rho \in |K|$; we set $|\infty| := \infty$ where the first ∞ is in $\mathbb{P}$ and the second ∞ is in $|K|_\infty$.

Disks. An *open disk* (in $\mathbb{P}$) is a set

$$D \ = \ \{z \in \mathbb{P} : \ |z-a| < \rho\}$$

or a set

$$D \ = \ \{z \in \mathbb{P} : \ |z-a| > \rho\}$$

with $a \in K,\ \rho \in |K^\times|$. Open disks of the first kind are subsets of K; those of the second kind are subsets of $\mathbb{P}$ containing the point ∞.

A *closed disk* (in $\mathbb{P}$) is a set

$$D \ = \ \{z \in \mathbb{P} : \ |z-a| \leqslant \rho\}$$

or a set

$$D \ = \ \{z \in \mathbb{P} : \ |z-a| \geqslant \rho\}$$

with $a \in K,\ \rho \in |K^\times|$. Closed disks of the first kind are subsets of K and those of the second kind are subsets of $\mathbb{P}$ containing the point ∞. By a *disk* we mean an open disk or a closed disk. A disk contained in K is also called a *bounded disk*, and a disk containing the point $\infty \in \mathbb{P}$ is called an *unbounded disk*. The closed disks are exactly the complements in $\mathbb{P}$ of the open disks. If the valuation is trivial (that is, $|K^\times| = \{1\}$), then the open disks are the one-element subsets of $\mathbb{P}$.

Note that for $a, b \in K$ and $\rho \in |K^\times|$ we have:

$$\begin{aligned}
|b-a| < \rho \ &\Rightarrow\ \{z \in \mathbb{P}:\ |z-a| < \rho\} \ = \ \{z \in \mathbb{P}:\ |z-b| < \rho\},\\
|b-a| < \rho \ &\Rightarrow\ \{z \in \mathbb{P}:\ |z-a| \geqslant \rho\} \ = \ \{z \in \mathbb{P}:\ |z-b| \geqslant \rho\},\\
|b-a| \leqslant \rho \ &\Rightarrow\ \{z \in \mathbb{P}:\ |z-a| \leqslant \rho\} \ = \ \{z \in \mathbb{P}:\ |z-b| \leqslant \rho\},\\
|b-a| \leqslant \rho \ &\Rightarrow\ \{z \in \mathbb{P}:\ |z-a| > \rho\} \ = \ \{z \in \mathbb{P}:\ |z-b| > \rho\}.
\end{aligned}$$

Thus for bounded disks D, E, either $D \cap E = \emptyset$, or $D \subseteq E$, or $E \subseteq D$. Note also that for a bounded open disk (respectively, bounded closed disk) D there is a unique $\rho \in |K^\times|$ such that for some $a \in K$ we have $D = \{z \in K:\ |z-a| < \rho\}$ (respectively, $D = \{z \in K:\ |z-a| \leqslant \rho\}$); set $\operatorname{radius}(D) := \rho$ for this particular ρ.

Lemma 11.1. *Let $g \in \mathrm{GL}_2(K)$ and let D be an open* (*respectively, closed*) *disk. Then gD is also an open* (*respectively, closed*) *disk. The resulting action of $\mathrm{GL}_2(K)$ on the set of open disks* (*respectively, closed disks*) *is transitive.*

Proof. It suffices to prove the claims about g for g a translation $z \mapsto z + a : \mathbb{P} \to \mathbb{P}$ (with $a \in K$), for g a dilation $z \mapsto az$ (with $a \in K^\times$), and for g the inversion $z \mapsto 1/z : \mathbb{P} \to \mathbb{P}$. The case of translations and dilations is clear, and each of the resulting actions of the affine subgroup of $\mathrm{GL}_2(K)$ on the set of bounded open disks, the set of bounded closed disks, the set of unbounded open disks, and the set of unbounded closed disks is transitive. Suppose next that g is the above inversion. If D is a bounded open disk with $0 \in D$, then $D = \{z \in K:\ |z| < \rho\}$ with $\rho \in |K^\times|$, so $gD = \{z \in \mathbb{P}:\ |z| > \rho^{-1}\}$ is an unbounded open disk with $0 \notin gD$. Likewise, if D is a bounded closed disk with $0 \in D$, then gD is an unbounded closed disk with $0 \notin gD$, and if D is an unbounded open (respectively, closed) disk with $0 \notin D$, then gD is a bounded open (respectively, closed) disk with $0 \in gD$. Next, if D is a bounded open disk with $0 \notin D$, then $D = \{z \in K:\ |z-a| < \rho\}$ with $a \in K, |a| \geqslant \rho \in |K^\times|$, so $gD = \{y \in K:\ |y - a^{-1}| < |a|^{-2}\rho\}$ is also a bounded open disk with $0 \notin gD$. Likewise, if D is a bounded closed disk with $0 \notin D$, then gD is also a bounded closed disk with $0 \notin gD$. Taking complements it follows that if D is an unbounded open (respectively, closed) disk with $0 \in D$, then so is gD. $\square$

Using this lemma and the corresponding result for bounded disks we conclude that for any disks D, E with $D \cup E \neq \mathbb{P}$, either $D \cap E = \emptyset$, or $D \subseteq E$, or $E \subseteq D$.

Affinoids. In the rest of this chapter we fix an algebraically closed field K^{a} equipped with a nontrivial valuation. We use multiplicative notation as before, and $\mathbb{P}$ and the various notions of disk are now with respect to the valued field K^{a}.

A *connected affinoid* is a nonempty set $F = \mathbb{P} \setminus (E_1 \cup \cdots \cup E_m)$ where $E_1, \ldots, E_m$ are open disks. Such F is said to be *bounded* if $\infty \notin F$. Suppose F is a bounded connected affinoid. Then $F = D \setminus (H_1 \cup \cdots \cup H_m)$ with D a bounded closed disk and $H_1, \ldots, H_m$ *disjoint* open disks contained in D, and this uniquely determines D (the ambient disk of F) and $\{H_1, \ldots, H_m\}$ (the set of holes of F). Conversely, for any bounded closed disk D and disjoint open disks $H_1, \ldots, H_m \subseteq D$, the set $F = D \setminus (H_1 \cup \cdots \cup H_m)$ is a connected affinoid. Closed disks are connected affinoids.

Using Lemma 11.1 we easily obtain: for any connected affinoid F there are disjoint open disks $E_1, \ldots, E_m$ such that $F = \mathbb{P} \setminus (E_1 \cup \cdots \cup E_m)$, and this determines uniquely the set $\{E_1, \ldots, E_m\}$. It is also easy to see that a connected affinoid is an infinite subset of $\mathbb{P}$. Here is another useful fact:

Lemma 11.2. *Let F_1 and F_2 be connected affinoids such that $F_1 \cap F_2 \neq \emptyset$. Then $F_1 \cup F_2$ and $F_1 \cap F_2$ are connected affinoids.*

An *affinoid* is a finite union of connected affinoids. Note that if F_1 and F_2 are affinoids, then so are $F_1 \cap F_2$ and $F_1 \cup F_2$. A *component* of an affinoid F is a maximal (under inclusion) connected affinoid contained in F.

Lemma 11.3. *Let F be an affinoid. Then the components of F are the elements of a finite partition of F.* (*The empty subset of $\mathbb{P}$ is an affinoid and has no components.*)

Proof. We have $F = F_1 \cup \cdots \cup F_n$ where the F_i are connected affinoids. Take the maximal subsets I of $\{1, \dots, n\}$ for which $F_I := \bigcup_{i \in I} F_i$ is a connected affinoid. Using Lemma 11.2 it is easy to check that the F_I for these maximal I are exactly the components of F, and that any two different components of F are disjoint. $\square$

The affinoids associated to a rational function. These affinoids are the subsets F of $\mathbb{P}$ in the following proposition:

Theorem 11.4. *Let $r(Z) \in K^{\mathrm{a}}(Z)^{\times}$, and $\rho \in |K^{\mathrm{a},\times}|$. Then*

$$F \ := \ \{z \in \mathbb{P} : \ |r(z)| \leqslant \rho\}$$

is an affinoid. (*This includes the possibility that $F = \emptyset$.*)

Proof. Multiplying r with an element of $K^{\mathrm{a},\times}$ we arrange that

$$r(Z) \ = \ \prod_{i=1}^{m} (Z - a_i)^{e_i}$$

with distinct $a_1, \dots, a_m \in K^{\mathrm{a}}$ and $e_1, \dots, e_m \in \mathbb{Z}^{\neq}$. The proposition holds clearly for $m = 0$ and $m = 1$, so assume $m \geqslant 2$. Let $i, j \in \{1, \dots, m\}$ below.

Special Case: $|a_i - a_j| = 1$ for all $i \neq j$.

SUBCASE (a): $\rho \geqslant 1$. Set

$$F_0 \ := \ \{z \in \mathbb{P} : \ |r(z)| \leqslant \rho, \ |z - a_i| \geqslant 1 \text{ for all } i\}.$$

Note that for $z \in \mathbb{P}$, if $|z - a_i| \geqslant 1$ for all i, then $|z - a_i| = |z - a_1|$ for all i. Thus for $e := e_1 + \cdots + e_m$ we have $F_0 = \{z \in \mathbb{P} : \ |(z - a_1)^e| \leqslant \rho, \ |z - a_i| \geqslant 1 \text{ for all } i\}$. There are $z \in K^{\mathrm{a}}$ with $|z - a_i| = 1$ for all i, so $F_0 \neq \emptyset$, and thus F_0 is a connected affinoid. Moreover, if $z \in \mathbb{P}$ and $|z - a_i| < 1$, then $|z - a_j| = 1$ for all $j \neq i$, so $|r(z)| = |(z - a_i)^{e_i}|$. Hence

$$F_i := \{z \in \mathbb{P} : \ |r(z)| \leqslant \rho \text{ and } |z - a_i| < 1\} \ = \ \{z \in \mathbb{P} : \ |(z - a_i)^{e_i}| \leqslant \rho, \ |z - a_i| < 1\},$$

Considering the various possible signs of e and e_i we see that the set $F_0 \cup F_i$ is a connected affinoid. Hence $F = F_0 \cup F_1 \cup \cdots \cup F_m$ is a connected affinoid.

SUBCASE (b): $\rho < 1$. As in Subcase (a) we see that

$$F_0 \ := \ \{z \in \mathbb{P} : \ |r(z)| \leqslant \rho, \ |z - a_i| \geqslant 1 \text{ for all } i\}$$

is a connected affinoid or empty, and that

$$F_i \;:=\; \{z \in \mathbb{P} : \; |r(z)| \leqslant \rho \text{ and } |z - a_i| < 1\}$$

is a connected affinoid or empty. It follows that F is an affinoid.

General Case. We arrange $\mu := \max_{i \neq j} |a_i - a_j| = \max_{j \neq 1} |a_1 - a_j|$ and set

$$B \;:=\; \{z \in K^{\mathrm{a}} : \; |z - a_1| < \mu\}.$$

SUBCASE (c): $|a_i - a_j| = \mu$ for all $i \neq j$. This subcase reduces to the Special Case considered by a change of variables. This includes the case $m = 2$. Assume we are not in Subcase (c). Then $m \geqslant 3$, and we can arrange:

SUBCASE (d): $s \in \{2, \ldots, m-1\}$ is such that $a_1, \ldots, a_s \in B$ and $a_{s+1}, \ldots, a_m$ are outside B. Take $d \in (K^{\mathrm{a}})^\times$ with $|a_1 - a_i| < |d| < \mu$ for $i = 1, \ldots, s$ and set

$$D \;:=\; \{z \in \mathbb{P} : \; |z - a_1| \leqslant |d|\}, \qquad F_1 \;:=\; \{z \in \mathbb{P} : \; |r(z)| \leqslant \rho, \; |z - a_1| \leqslant |d|\}.$$

Then we have

$$F_1 \;=\; \{z \in \mathbb{P} : \; |(z - a_1)^{e_1} \cdots (z - a_s)^{e_s}| \leqslant \frac{\rho}{\mu^{e_{s+1} + \cdots + e_m}}\} \cap D.$$

We also set

$$E \;:=\; \{z \in \mathbb{P} : \; |z - a_1| \geqslant |d|\}, \qquad F_2 \;:=\; \{z \in \mathbb{P} : \; |r(z)| \leqslant \rho, \; |z - a_1| \geqslant |d|\}.$$

Then $F = F_1 \cup F_2$, and we have

$$F_2 \;:=\; \{z \in \mathbb{P} : \; |(z - a_1)^{e_1 + \cdots + e_s} \cdot (z - a_{s+1})^{e_{s+1}} \cdots (z - a_m)^{e_m}| \leqslant \rho\} \cap E.$$

One can assume inductively that F_1 and F_2 are affinoids. Thus F is an affinoid. $\square$

K-affinoids. The proof of Theorem 11.4 has some useful consequences that involve a subfield K of K^{a}. By a *connected K-affinoid* we mean a nonempty set

$$\mathbb{P} \setminus (E_1 \cup \cdots \cup E_m)$$

where each E_i is an open disk $\{z \in \mathbb{P} : \; |z - a| < \rho\}$ or $\{z \in \mathbb{P} : \; |z - a| > \rho\}$ with $a \in K$ and $\rho \in |K^\times|$. Of course, connected K-affinoids are connected affinoids. If F_1, F_2 are connected K-affinoids with $F_1 \cap F_2 \neq \emptyset$, then so are $F_1 \cup F_2$ and $F_1 \cap F_2$. For $a, b_1, \ldots, b_m \in K$ and $\rho, \rho_1, \ldots, \rho_m \in |K^\times|$ the set

$$\{z \in \mathbb{P} : \; |z - a| \leqslant \rho, \; |z - b_1| \geqslant \rho_1, \ldots, |z - b_m| \geqslant \rho_m\}$$

is empty or a bounded connected K-affinoid; every bounded connected K-affinoid has this form.

Corollary 11.5. *Let K be an algebraically closed subfield of K^{a} (possibly with $|K^\times| = \{1\}$), and suppose $r(Z) \in K(Z)^\times$, and $\rho \in |K|^\times$. Then the affinoid*

$$F \;:=\; \{z \in \mathbb{P} : \; |r(z)| \leqslant \rho\}$$

is a finite union of connected K-affinoids.

Proof. We follow the proof of Proposition 11.4. Note that if $m \geqslant 2$ and $|K^\times| = \{1\}$, then we are in the Special Case with $\rho = 1$, so in Subcase (a), and hence F_0 and $F_0 \cup F_i$ for $i \in \{1, \dots, m\}$ are connected K-affinoids. $\square$

The following variant will also be useful:

Corollary 11.6. *Let F be a connected affinoid and suppose the rational functions $r_1, r_2 \in K^{\mathrm{a}}(Z)$ have no pole in F. Then $\{z \in F : \ |r_1(z)| \leqslant |r_2(z)|\}$ is a finite union of connected affinoids and singletons, all contained in F.*

If in addition K is an algebraically closed subfield of K^{a} and $r_1, r_2 \in K(Z)$, then $\{z \in F : \ |r_1(z)| \leqslant |r_2(z)|\}$ is a finite union of connected K-affinoids and singletons $\{z\}$ with $z \in K$, all contained in F.

Proof. The case $r_2 = 0$ is trivial, so assume $r_2 \neq 0$ and set $r(Z) := r_1(Z)/r_2(Z)$. One verifies easily that then for $z \in F$ we have

$$
\begin{aligned}
|r(z)| \leqslant 1 &\implies |r_1(z)| \leqslant |r_2(z)|, \\
|r_1(z)| \leqslant |r_2(z)| &\implies |r(z)| \leqslant 1 \text{ or } v_z(r_1) \geqslant v_z(r_2) > 0.
\end{aligned}
$$

It remains to note that $\{z \in F : \ v_z(r_2) > 0\} = \{z \in F : r_2(z) = 0\}$ is finite. $\square$

As before, K is a valued subfield of K^{a}. It can happen that a connected K-affinoid has empty intersection with K even when the valuation is nontrivial on K. As an example, let p be a prime number, take the p-adic field $K = \mathbb{Q}_p$ with its usual p-adic absolute value, take an algebraically closed valued field extension K^{a} of K, and take $F = D - (H_1 \cup \cdots \cup H_p)$ where

$$
D := \{z \in K^{\mathrm{a}} : \ |z| \leqslant 1\}, \quad H_i := \{z \in K^{\mathrm{a}} : \ |z - i| \leqslant p^{-1}\} = pD + i \quad (i = 1, \dots, p).
$$

Then $D \cap K = \mathbb{Z}_p,\ H_i \cap K = p\mathbb{Z}_p + i$ for $i = 1, \dots, p$, so $F \cap \mathbb{Q}_p = \emptyset$, although F is a (bounded) connected K-affinoid. On the other hand:

Lemma 11.7. *Let F be a bounded connected K-affinoid with m holes, and suppose the valuation on K is nontrivial and the residue field of K has more than m elements. Then F contains an entire disk $\{z \in K^{\mathrm{a}} : \ |z - a| < \rho\}$ with a in K and $\rho \in |K^\times|$, so $F \cap K$ is infinite.*

Proof. By translation and dilation we reduce to the case $F = D - (H_1 \cup \cdots \cup H_m)$ where $D = \{z \in K^{\mathrm{a}} : \ |z| \leqslant 1\}$ is the ambient disk of F and $H_1, \dots, H_m$ are the holes of F. For $i = 1, \dots, m$ we have $H_i \subseteq \{z \in K^{\mathrm{a}} : \ |z - a_i| < 1\}$ with $a_i \in D \cap K$. Taking $a \in K$ with $|a| \leqslant 1$ and $|a - a_i| = 1$ for $i = 1, \dots, m$ we have $\{z \in K^{\mathrm{a}} : \ |z - a| < 1\} \subseteq F$. $\square$

R-affinoids. We continue with the subfield K of K^{a} and let $R := K \cap R^{\mathrm{a}}$ be the valuation ring of K as a *valued* subfield of K^{a}. Define a *closed* (respectively *open*) R-disk to be a subset of K^{a} of the form

$$
\{z \in K^{\mathrm{a}} : \ |z - c| \leqslant |\pi|\}, \quad \text{respectively,} \quad \{z \in K^{\mathrm{a}} : \ |z - c| < |\pi|\},
$$

where $c, \pi \in R$, $\pi \neq 0$. So closed (respectively, open) R-disks are closed (respectively, open) disks in the sense of the ambient algebraically closed valued field K^{a} and are subsets of R^{a}. Likewise, a *connected R-affinoid* is a set $D \setminus (H_1 \cup \cdots \cup H_m)$ where D is a closed R-disk and $H_1, \dots, H_m$ are disjoint open R-disks contained in D. Thus a connected R-affinoid is a connected affinoid and a subset of R^{a}. Moreover, for a connected affinoid F we have:

$$F \text{ is a connected } R\text{-affinoid} \iff F \subseteq R^{\mathrm{a}} \text{ and } F \text{ is a connected } K\text{-affinoid}.$$

If F_1 and F_2 are connected R-affinoids and $F_1 \cap F_2 \neq \emptyset$, then $F_1 \cup F_2$ and $F_1 \cap F_2$ are also connected R-affinoids. The next result will not be used, but may give the reader some feeling for the nature of certain kinds of connected R-affinoids.

Corollary 11.8. *Suppose F is a connected R-affinoid, $z \in F$, and $\overline{z} \notin \boldsymbol{k}$. Then there are $M \in \mathbb{N}$ and $a_1, \dots, a_M \in R$ such that*

$$F \supseteq \{u \in R^{\mathrm{a}} : \overline{u} \neq \overline{a}_1, \dots, \overline{a}_M\}.$$

Proof. Let D be the ambient disk of F. Since $D \subseteq R^{\mathrm{a}}$ we have

$$D = \{y \in K^{\mathrm{a}} : |y - c| \leqslant |\pi|\} \text{ where } c, \pi \in R,\ \pi \neq 0.$$

Then $\overline{z} \neq \overline{c}$, so $|z - c| = 1$, and thus $|\pi| = 1$. Hence $D = R^{\mathrm{a}}$. It remains to note that each hole of F is contained, for some $a \in R$, in the open disk

$$\{y \in K^{\mathrm{a}} : |y - a| < 1\} = \{y \in R^{\mathrm{a}} : \overline{y} = \overline{a}\} \qquad \square$$

11.2 Affinoid algebras

In this section we introduce here affinoid algebras on suitable affinoids, inspired by [37]. We are forced, however, to replace the simple analytic definitions and proofs in [37] by more elaborate constructions and arguments that depend heavily on Section 10.3. This is because the way we develop our AKE-theory for suitable valuation A-rings in Sections 12.1 and 12.2 requires a first-order setting.

Here we return to the assumptions of Sections 10.3: *A is noetherian with an ideal $\mathcal{o}(A) \neq A$ such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ and A is $\mathcal{o}(A)$-adically complete; R is a viable valuation A-ring.*

We fix $t \in R$ such that $\mathcal{o}(R) = tR$ and adopt the notations and terminology concerning R and its fraction field K from Section 10.3, with the valuation $v : K^\times \to \Gamma$ on K such that $R = \{a \in K : va \geqslant 0\}$, so vt is the least positive element of Γ. We shall also need the extension $\widetilde{A} = A\langle \boldsymbol{\omega} \rangle[[t_p]]$ of A at a few places in this section. Recall K^{a} is an algebraically closed A-extension of K whose valuation A-ring is denoted by R^{a}.

In the rest of this section F is a connected R-affinoid. Then we have $n \geqslant 1$ and $c_1, \dots, c_n, \pi_1, \dots, \pi_n \in R$ with $\pi_1, \dots, \pi_n \neq 0$ such that

$$F = \{z \in R^{\mathrm{a}} : |z - c_1| \leqslant |\pi_1|,\ |z - c_2| \geqslant |\pi_2|, \dots, |z - c_n| \geqslant |\pi_n|\}$$

where the open R-disks $\{z \in R^{\mathrm{a}} : |z - c_i| < |\pi_i|\}$, for $i = 2, \dots, n$, are pairwise disjoint and contained in the closed R-disk $\{z \in R^{\mathrm{a}} : |z - c_1| \leqslant |\pi_1|\}$. We express this by: F is given by $(c_1, \dots, c_n; \pi_1, \dots, \pi_n)$; we assume this till further notice. Define $\psi : F \to (R^{\mathrm{a}})^n$ by

$$\psi(z) := \left(\frac{z - c_1}{\pi_1}, \frac{\pi_2}{z - c_2}, \dots, \frac{\pi_n}{z - c_n}\right).$$

For distinct $i, j \in \{2, \ldots, n\}$, set $s_{ij} := \pi_i/(c_i - c_j)$, and for $j = 2, \ldots, n$, set $s_{1j} := (c_j - c_1)/\pi_1$, and $s_{j1} := \pi_j/\pi_1$. So $|s_{ij}| \leqslant 1$ for all distinct $i, j \in \{1, \ldots, n\}$. We also introduce the polynomials $e_{ij} \in R[Y]$ for $1 \leqslant i < j \leqslant n$:

$$
\begin{aligned}
e_{1j}(Y) &:= Y_1 Y_j - s_{1j} Y_j - s_{j1} \qquad (2 \leqslant j \leqslant n) \\
e_{ij}(Y) &:= Y_i Y_j + s_{ji} Y_i + s_{ij} Y_j \qquad (2 \leqslant i < j \leqslant n).
\end{aligned}
$$

Let $I(F)$ be the ideal of $K\langle Y \rangle$ generated by the e_{ij} with $1 \leqslant i < j \leqslant n$. Then

$$
\psi(F) = \mathrm{Z}\big(I(F)\big) := \{y \in (R^{\mathrm{a}})^n : f(y) = 0 \text{ for all } f \in I(F)\}.
$$

For the inclusion from right to left, let $y = (y_1, \ldots, y_n) \in \mathrm{Z}\big(I(F)\big)$. Then $y_1 = (z - c_1)/\pi_1$ with $z \in R^{\mathrm{a}}$, $|z - c_1| \leqslant |\pi_1|$. For $j = 2, \ldots, n$, $e_{1j}(y) = 0$ gives $z \neq c_j$, $y_j = \pi_j/(z - c_j)$, so $y_j \in R^{\mathrm{a}}$ gives $z \in F$ and thus $y = \psi(z) \in \psi(F)$.

Lemma 11.9. *Let $g \in K\langle Y \rangle$ in* (i)*, and $n \geqslant 2$, $g \in K\langle Y_2, \ldots, Y_n \rangle$ in* (ii)*. Then:*

(i) $g \equiv g_1 + \cdots + g_n \mod I(F)$ *for some $g_j \in K\langle Y_j \rangle$, $j = 1, \ldots, n$;*

(ii) $g \equiv g_2 + \cdots + g_n \mod I_1$ *for some $g_j \in K\langle Y_j \rangle$, $j = 2, \ldots, n$, where I_1 is the ideal of $K\langle Y_2, \ldots, Y_n \rangle$ generated by the e_{ij} with $2 \leqslant i < j \leqslant n$.*

Proof. For (i) we use induction on n. The case $n = 1$ is obvious, so assume $n \geqslant 2$. Let I' be the ideal of $K\langle Y' \rangle$ generated by the e_{ij} with $1 \leqslant i < j \leqslant n-1$. Assume inductively that each $g' \in K\langle Y' \rangle$ is congruent modulo I' to a sum $g'_1 + \cdots + g'_{n-1}$ with $g'_i = g'_i(Y_i) \in K\langle Y_i \rangle$. Then (i) follows from what we prove next:

Claim: $g \equiv h_1 + h_2 \mod I(F)$ for some $h_1 \in K\langle Y' \rangle$ and $h_2 \in K\langle Y_n \rangle$.

Multiplying g by an element of $K^{\times}$ we arrange $g \in R\langle Y \rangle$, so $g = G(x, Y)$ with $G = \sum_\nu a_\nu(X) Y^\nu \in A\langle X, Y \rangle$ and $x \in R^m$ (and $a_\nu(X) \to 0$ as $|\nu| \to \infty$). Let

$$
S := (S_{1n}, S_{n1}, \ldots, S_{in}, S_{ni}, \ldots, S_{n-1,n}, S_{n,n-1})
$$

be a fresh tuple of indeterminates, and in the polynomial ring $A[S, Y]$, set

$$
\begin{aligned}
E_{1n} &:= Y_1 Y_n - S_{n1} - S_{1n} Y_n \\
E_{in} &:= Y_i Y_n + S_{ni} Y_i + S_{in} Y_n \quad (2 \leqslant i \leqslant n-1).
\end{aligned}
$$

Working modulo the ideal $(E_{1n}, \ldots, E_{n-1,n})$ in $A[S, Y]$ an easy induction on $|\nu|$ gives for every $\nu \in \mathbb{N}^n$ polynomials $p^{(\nu)}(S, Y') \in A[S, Y'] \subseteq A[S, Y]$ and $q^{(\nu)}(S, Y_n)$ in $A[S, Y_n] \subseteq A[S, Y]$ such that

$$
\begin{aligned}
Y^\nu &\equiv p^{(\nu)} + q^{(\nu)} \mod (E_{1n}, \ldots, E_{n-1,n}), \quad \deg_{Y'} p^{(\nu)}, \deg_{Y_n} q^{(\nu)} \leqslant |\nu|, \text{ so} \\
Y^\nu &= p^{(\nu)} + q^{(\nu)} + \sum_{i=1}^{n-1} r_i^{(\nu)} E_{in}, \quad r_1, \ldots, r_{n-1} \in A[S, Y].
\end{aligned}
$$

Now we combine these equalities in $A\langle X,S,Y\rangle$ and set

$$\begin{aligned}
H_1(X,S,Y') &:= \sum_\nu a_\nu(X)p^{(\nu)}(S,Y') \in A\langle X,S,Y'\rangle \subseteq A\langle X,S,Y\rangle \\
H_2(X,S,Y_n) &:= \sum_\nu a_\nu(X)q^{(\nu)}(S,Y_n) \in A\langle X,S,Y_n\rangle \subseteq A\langle X,S,Y\rangle \\
R_i(X,S,Y) &:= \sum_\nu a_\nu(X)r_i^{(\nu)}(S,Y) \in A\langle X,S,Y\rangle \qquad (i=1,\dots,n-1), \text{ so} \\
\sum_\nu a_\nu(X)Y^\nu &= H_1(X,S,Y') + H_2(X,S,Y_n) + \sum_{i=1}^{n-1} R_i(X,S,Y)E_{in}.
\end{aligned}$$

For $s := (s_{1n}, s_{n1}, \dots, s_{in}, s_{ni}, \dots, s_{n-1,n}, s_{n,n-1})$ we have $e_{in} = E_{in}(s,Y)$, so

$$g(Y) \ = \ G(x,Y) \ = \ H_1(x,s,Y') + H_2(x,s,Y_n) + E(x,s,Y)$$

with $E(x,s,Y) \in (e_{1n},\dots,e_{n-1,n})R\langle Y\rangle \subseteq I(F)$.

The proof of (ii) is similar, and we only indicate the beginning. The case $n=2$ is trivial, so assume $n \geqslant 3$. Let I_1' be the ideal of $K\langle Y_2,\dots,Y_{n-1}\rangle$ generated by the e_{ij} with $2 \leqslant i < j \leqslant n-1$. Assume inductively that each $g' \in K\langle Y_2,\dots,Y_{n-1}\rangle$ is congruent modulo I_1' to a sum $g_2' + \dots + g_{n-1}'$ with $g_i' = g_i'(Y_i) \in K\langle Y_i\rangle$ for $i = 2,\dots,n-1$. Then (ii) follows from

Claim: $g \equiv h_1 + h_2 \mod I_1$ for some $h_1 \in K\langle Y_2,\dots,Y_{n-1}\rangle$ and $h_2 \in K\langle Y_n\rangle$.

The proof of this claim is just like that of the claim in the proof of (i). ☐

For $i > j$ in $\{2,\dots,n\}$ we set $e_{ij} := e_{ji}$.

Lemma 11.10. *Let $g \in K\langle Y\rangle$. Then $Pg \equiv EQ \mod I(F)$ for some unit E of $R\langle Y\rangle$, polynomial $Q \in K[Y_k]$ with $k \in \{1,\dots,n\}$, and polynomial $P \in R[Y]$ which is a product of finitely many factors, each of the form $Y_j + s_{ji}$ $(2 \leqslant i,j \leqslant n, i \neq j)$ or of the form $Y_1 - s_{1i}$ $(2 \leqslant i \leqslant n)$, or of the form Y_j $(2 \leqslant j \leqslant n)$.*

Proof. If $g \in I(F)$, this holds with $P = E = 1$ and $Q = 0$, so assume $g \notin I(F)$ below. Lemma 11.9(i) gives $g_i \in K\langle Y_i\rangle$ for $i = 1,\dots,n$ such that $g \equiv g_1 + \dots + g_n \mod I(F)$, and we can arrange $g = g_1 + \dots + g_n$. Then Proposition 10.38 gives $i \in \{1,\dots,n\}$, $\ell \in \mathbb{N}$, $c \in K^\times$, a unit E of $R\langle Y\rangle$, and $c_1,\dots,c_\ell \in R\langle Y^*\rangle$ where $Y^* := (Y_1,\dots,Y_{i-1},Y_{i+1},\dots,Y_n)$, such that

$$g \ = \ cE \cdot \big(Y_i^\ell + c_1 Y_i^{\ell-1} + \dots + c_\ell\big).$$

For $n = 1$ the desired property holds with $P = 1$. Below we assume $n \geqslant 2$. Pick $j \in \{1,\dots,n\}$, $j \neq i$. Suppose $i,j > 1$ for now. Since $e_{ij} - s_{ij}Y_j = Y_i(s_{ji} + Y_j)$ we can multiply both sides in the last display by $(s_{ji} + Y_j)^\ell$ to obtain

$$g(s_{ji}+Y_j)^\ell \ = \ cE \cdot \big((e_{ij} - s_{ij}Y_j)^\ell + c_1(e_{ij} - s_{ij}Y_j)^{\ell-1}(s_{ji}+Y_j) + \dots + c_\ell(s_{ji}+Y_j)^\ell\big).$$

We have $e_{ij} \in I(F)$, so for

$$h(Y^*) \ := \ (-s_{ij}Y_j)^\ell + c_1(-s_{ij}Y_j)^{\ell-1}(s_{ji}+Y_j) + \dots + c_l(s_{ji}+Y_j)^\ell \in K\langle Y^*\rangle$$

we obtain

$$(s_{ji}+Y_j)^\ell \cdot g \ \equiv \ cEh \quad \bmod I(F).$$

For $j=1$ we use the identity $e_{1i}+s_{i1}=Y_i(Y_1-s_{1i})$ to introduce instead of a factor $(s_{ji}+Y_j)^\ell$ a factor $(Y_1-s_{1i})^\ell$ and obtain a similar congruence for some $h\in K\langle Y^*\rangle$; if $i=1$, we use $Y_1Y_j=e_{1j}+s_{j1}+s_{1j}Y_j$ to introduce likewise a factor Y_j^ℓ.

We can assume inductively that the lemma holds for $h\in K\langle Y^*\rangle$ instead of $g\in K\langle Y\rangle$, with the ideal of $K\langle Y^*\rangle$ generated by the e_{kl} $(1\leqslant k<l\leqslant n,\ k,l\neq i)$ instead of $I(F)$. (In case $i=1$ this involves an argument as above using Lemma 11.9(ii) instead of Lemma 11.9(i).) This yields the desired conclusion. $\square$

Let $\mathcal{R}(F)$ be the K-algebra of K^{a}-valued functions on F. Then

$$\psi^*: K\langle Y\rangle \to \mathcal{R}(F), \quad \psi^*(g)(z) := g\big(\psi(z)\big) \text{ for } g\in K\langle Y\rangle,\ z\in F,$$

is a K-algebra morphism, and we set $\mathcal{O}(F) := \psi^*(K\langle Y\rangle)$, a K-subalgebra of $\mathcal{R}(F)$. It is clear that $I(F)\subseteq \ker(\psi^*)$, and in fact we have equality here: Corollary 11.19. Note also that for $f\in\mathcal{O}(F)$ and $z\in F\cap R$ we have $f(z)\in K$. First we show how elements of $\mathcal{O}(F)$ relate to rational functions:

Corollary 11.11. *Given $f\in\mathcal{O}(F)$, there is a unit $E\in R\langle Y\rangle$ and a rational function $r(Z)\in K(Z)$ without poles in F such that $f(z)=\psi^*(E)(z)\cdot r(z)$ for all $z\in F$. For such f, E, r, we have $D(Y)\in R[Y]$ with $E-D\in o(A)R\langle Y\rangle$, and then*

(i) $|f(z)|=|r(z)|$ for all $z\in F$;

(ii) $f(z)\sim D\big(\psi(z)\big)\cdot r(z)$ for all $z\in F$ with $f(z)\neq 0$;

(iii) there exists $\rho(Z)\in K(Z)$ without poles in F such that for all $z\in F$ with $f(z)\neq 0$ we have $f(z)\sim\rho(z)$.

Proof. Let $g\in K\langle Y\rangle$ and $f=\psi^*(g)$. Let $E\in R\langle Y\rangle$, $Q\in K[Y_k]$, and $P\in R[Y]$ be as in Lemma 11.10. Then for $z\in F$ we have

$$\begin{aligned}
\psi^*(Y_j+s_{ji})(z) &= \frac{\pi_j(z-c_i)}{(c_j-c_i)(z-c_j)} && (2\leqslant i,j\leqslant n,\ i\neq j),\\
\psi^*(Y_1-s_{1i})(z) &= \frac{z-c_i}{\pi_1} && (2\leqslant i\leqslant n),\\
\psi^*(Y_j)(z) &= \frac{\pi_j}{z-c_j} && (2\leqslant j\leqslant n).
\end{aligned}$$

Hence $\psi^*(P)(z)\neq 0$ for all $z\in F$. Take $r(Z)\in K(Z)$ such that for all $z\in F$ we have $r(z)=\psi^*(Q)(z)/\psi^*(P)(z)$. Then $r(Z)$ has no poles in F and $f(z)=\psi^*(E)(z)\cdot r(z)$ for all $z\in F$. The rest is routine. $\square$

We used $(c_1,\dots,c_n;\pi_1,\dots,\pi_n)$ to define $\mathcal{O}(F)$ as a subring of $\mathcal{R}(F)$. Fortunately:

Lemma 11.12. *The ring $\mathcal{O}(F)$ does not depend on $(c_1,\dots,c_n;\pi_1,\dots,\pi_n)$.*

Proof. Suppose F is also given by $(c_1',\dots,c_n';\pi_1',\dots,\pi_n')$. We first consider the case $c_1=c_1'$, $\pi_1=\pi_1'$ and σ is a permutation of $\{2,\dots,n\}$ such that $c'_{\sigma(j)}=c_j$ and $\pi'_{\sigma(j)}=\pi_j$ for $j=2,\dots,n$. Let $\psi': F\to(R^{\mathrm{a}})^n$ be the map associated to $(c_1',\dots,c_n';\pi_1',\dots,\pi_n')$. Then $\psi(z)=\sigma(\psi'(z))$, where for $y\in(R^{\mathrm{a}})^n$ we set

$\sigma(y) = \big(y_1, y_{\sigma(2)}, \ldots, y_{\sigma(n)}\big)$. Thus $\mathcal{O}(F)$ as defined via ψ equals $\mathcal{O}(F)$ as defined via ψ'. Using this special case we may assume for the general case that for $j = 1, \ldots, n$ the R-disks

$$\{z \in K^{\mathrm{a}} : \ |z - c_j| < |\pi_j|\} \ \text{ and } \ \{z \in K^{\mathrm{a}} : \ |z - c_j'| < |\pi_j'|\}$$

are equal. Let $f \in \mathcal{O}(F)$, $G(X,Y) = \sum a_\nu(X) Y^\nu \in A\langle X, Y\rangle$, and let $x \in R^m$ be such that $f = \psi^*(g)$, where $g(Y) = G(x, Y) \in R\langle Y\rangle$. Set

$$v_1 \ := \ \frac{\pi_1'}{\pi_1}, \quad u_1 \ := \ \frac{c_1 - c_1'}{\pi_1'}, \quad v_j \ := \ \frac{\pi_j}{\pi_j'}, \quad u_j \ := \ \frac{c_j - c_j'}{t\pi_j'} \quad \text{ for } \ j = 2, \ldots, n,$$

so $u_1, \ldots, u_n, v_1, \ldots, v_n \in R$. We introduce distinct indeterminates

$$U_1, \ldots, U_n, V_1, \ldots, V_n, W_1, \ldots, W_n$$

different also from $X_1, X_2, \ldots, X_m$. Now

$$\psi^*(Y_1)(z) \ = \ \frac{z - c_1}{\pi_1} \ = \ v_1 \cdot \big(\frac{z - c_1'}{\pi_1'} - u_1\big) \ \text{ for } z \in F,$$

and for $j = 2, \ldots, n$ and $z \in F$,

$$\begin{aligned} \psi^*(Y_j)(z) \ &= \ \frac{\pi_j}{z - c_j} \ = \ \frac{\pi_j}{(z - c_j')} \frac{1}{\big(1 - (c_j - c_j')/(z - c_j')\big)} \\ &= \ v_j \cdot \big(\pi_j'/(z - c_j')\big) \cdot \frac{1}{1 - t u_j \big(\pi_j'/(z - c_j')\big)} \ = \ Q\big(u_j, v_j, \pi_j'/(z - c_j')\big) \end{aligned}$$

where $Q(U_j, V_j, W_j) := V_j \sum_{k=0}^\infty t_p^k U_j^k W_j^{k+1} \in \widetilde{A}\langle U_j, V_j, W_j\rangle$. (Here we use the identity $(1 - t_p U_j W_j) Q(U_j, V_j, W_j) = V_j W_j$.) Let $\psi'^* : K\langle W\rangle \to \mathcal{R}(F)$ be given by $\psi'^*(h)(z) = h\big(\psi'(z)\big)$ for $z \in F$, in particular, for $z \in F$,

$$\psi'^*(W_1)(z) \ = \ \frac{z - c_j'}{\pi_1'}, \qquad \psi'^*(W_j)(z) \ = \ \frac{\pi_j'}{z - c_j'} \ \text{ for } \ j = 2, \ldots, n.$$

With $U := (U_1, \ldots, U_n)$ and likewise with V, W, define $H \in \widetilde{A}\langle X, U, V, W\rangle$ by

$$H \ := \ G\big(X, V_1(W_1 - U_1), V_2 \sum_{k=0}^\infty (t_p U_2)^k W_2^{k+1}, \ldots, V_n \sum_{k=0}^\infty (t_p U_n)^k W_n^{k+1}\big).$$

Set $u := (u_1, \ldots, u_n) \in R^n$, $v := (v_1, \ldots, v_n) \in R^n$ and $h(W) := H(x, u, v, W)$ in $R\langle W\rangle$. Then for $z \in F$ we have $h\big(\psi'(z)\big) = g\big(\psi(z)\big)$, and so $\psi'^*(h) = f$. $\square$

Example. The connected R-affinoid $F := \{z \in K^{\mathrm{a}} : \ |z| \leqslant 1\} = R^{\mathrm{a}}$ is given by $(0; 1)$. The corresponding map $\psi : F \to R^{\mathrm{a}}$ is the identity map, so $\mathcal{O}(F) = K\langle Z\rangle$, where Z is a single indeterminate and where we identify each $g \in K\langle Z\rangle$ with the function $z \mapsto g(z) : F \to K^{\mathrm{a}}$ in $\mathcal{O}(F)$.

Lemma 11.13. *Let F' be a connected R-affinoid with $F' \subseteq F$. Then*

$$f \in \mathcal{O}(F) \implies f|_{F'} \in \mathcal{O}(F').$$

Proof. We can reduce to two cases: (1), where F' is obtained from F by adding a hole, possibly swallowing some holes of F, and (2), where we shrink the ambient disk $\{z \in R^{\mathrm{a}} : \ |z - c_1| \leqslant |\pi_1|\}$, possibly also eliminating some holes of F.

More precisely, in the first case F' is given for a certain $r \in \{1, \dots, n\}$ by

$$(c_1, c, c_{r+1}, \dots, c_n;\ \pi_1, \pi, \pi_{r+1}, \dots, \pi_n)$$

where $c, \pi \in R$, $\pi \neq 0$, and $\{z \in R^{\mathrm{a}} : \ |z - c| < |\pi|\}$ includes $\{z \in R^{\mathrm{a}} : \ |z - c_j| < |\pi_j|\}$ for $j = 2, \dots, r$ and is disjoint from $\{z \in R^{\mathrm{a}} : \ |z - c_j| < |\pi_j|\}$ for $j = r+1, \dots, n$. Let $f \in \mathcal{O}(F)$ with $f = \psi^*(g)$, $g \in R\langle Y \rangle$ and take $G(X, Y) \in A\langle X, Y \rangle$ such that $g(Y) = G(x, Y)$, $x \in R^m$. We show that then $f|_{F'} \in \mathcal{O}(F')$. Let

$$U_2, \dots, U_r, V_2, \dots, V_r, W_1, \dots, W_{n-r+2}$$

be distinct indeterminates also different from $X_1, \dots, X_m$ and set $U := (U_2, \dots, U_r)$ and likewise with V and W. Let $\psi' : F' \to (R^{\mathrm{a}})^{n-r+2}$ be the map associated to $(c_1, c, c_{r+1}, \dots, c_n;\ \pi_1, \pi, \pi_{r+1}, \dots, \pi_n)$. Set $u_j := (c_j - c)/t\pi$ and $v_j := \pi_j/\pi$ for $j = 2, \dots, r$, so $u := (u_2, \dots, u_r) \in R^{r-1}$ and $v := (v_2, \dots, v_r) \in R^{r-1}$. Let $H \in \widetilde{A}\langle X, U, V, W \rangle$ be

$$G\big(X, W_1, V_2 \sum_{k=0}^{\infty} (t_p U_2)^k W_2^{k+1}, \dots, V_r \sum_{k=0}^{\infty} (t_p U_r)^k W_2^{k+1}, W_3, \dots, W_{n-r+2}\big).$$

Then $h(W) := H(x, u, v, W)$ gives ${\psi'}^*(h) = f|_{F'}$ as in the proof of Lemma 11.12.

In the second case we assume F' is given for a certain $r \in \{1, \dots, n\}$ by

$$(c_1, c_{r+1}, \dots, c_n;\ \pi, \pi_{r+1}, \dots, \pi_n) \quad \text{where } \pi \in R^{\neq},\ |\pi| < |\pi_1|,$$

$\{z \in R^{\mathrm{a}} : \ |z - c_1| \leqslant |\pi|\}$ is disjoint from $\{z \in R^{\mathrm{a}} : \ |z - c_j| < |\pi_j|\}$ for $j = 2, \dots, r$ and includes $\{z \in R^{\mathrm{a}} : \ |z - c_j| < |\pi_j|\}$ for $j = r+1, \dots, n$. Let $f \in \mathcal{O}(F)$ with $f = \psi^*(g)$, $g \in R\langle Y \rangle$ and take $G(X, Y) \in A\langle X, Y \rangle$ and $x \in R^m$ such that $g(Y) = G(x, Y)$. We show that then $f|_{F'} \in \mathcal{O}(F')$. Let $\psi' : F' \to (R^{\mathrm{a}})^{n-r+1}$ be the map associated to $(c_1, c_{r+1}, \dots, c_n;\ \pi, \pi_{r+1}, \dots, \pi_n)$. For $j = 2, \dots, r$, $z \in F'$,

$$\frac{\pi_j}{z - c_j} = \frac{\pi_j}{c_1 - c_j} \cdot \frac{1}{1 - \left(\frac{z - c_1}{c_j - c_1}\right)} = \frac{\pi_j}{c_1 - c_j} \cdot \frac{1}{1 - \left(\frac{\pi}{c_j - c_1} \frac{z - c_1}{\pi}\right)} = \frac{v_j}{1 - t u_j \frac{z - c_1}{\pi}}$$

with $u_j := \frac{\pi}{\tau(c_j - c_1)} \in R$ and $v_j := \frac{\pi_j}{c_1 - c_j} \in R$. We also set $v_1 := \frac{\pi}{\pi_1} \in R$. Let

$$V_1, \dots, V_r, U_2, \dots, U_r$$

be distinct indeterminates, different also from $X_1, \dots, X_m, Y_1, \dots, Y_n$ and set $U := (U_2, \dots, U_r)$, $V := (V_1, \dots, V_r)$, and let $H \in \widetilde{A}\langle X, U, V, Y_1, Y_{r+1}, \dots, Y_n \rangle$ be

$$G(X, V_1 Y_1, V_2 \sum_{k=0}^{\infty} (t_p U_2)^k Y_1^k, \dots, V_r \sum_{k=0}^{\infty} (t_p U_r)^k Y_1^k, Y_{r+1}, \dots, Y_n).$$

Then for $u := (u_2, \ldots, u_r)$, $v := (v_1, \ldots, v_r)$, and

$$h(Y_1, Y_{r+1}, \ldots, Y_n) \ := \ H(x, u, v, Y_1, Y_{r+1}, \ldots, Y_n) \in R\langle Y_1, Y_{r+1}, \ldots, Y_n\rangle$$

we have $h\big(\psi'(z)\big) = g\big(\psi(z)\big) = f(z)$ for $z \in F'$. $\square$

Divisibility in $\mathcal{O}(F)$. We continue with our connected R-affinoid F and first deduce some easy consequences of the material in the preceding subsection. Here is one: let Z be a single variable and $g(Z) \in K\langle Z\rangle$; then by Lemma 11.13 and the example preceding it the function $z \mapsto g(z) : F \to K^{\mathrm{a}}$ belongs to $\mathcal{O}(F)$.

Corollary 11.14. *Let $f \in \mathcal{O}(F)$, $f \neq 0$. Then:*

(i) f has only finitely many zeros in F;

(ii) if f has no zeros in F, then f is a unit of $\mathcal{O}(F)$.

Proof. Item (i) is immediate from Corollary 11.11. For (ii), take $g \in K\langle Y\rangle$ with $f = \psi^*(g)$. Suppose f is not a unit of $\mathcal{O}(F)$. Then $1 \notin gK\langle Y\rangle + I(F)$, so by Theorem 10.39(iii) we have $y \in (R^{\mathrm{a}})^n$ such that $g(y) = 0$ and $h(y) = 0$ for all $h \in I(F)$. Then $y \in \mathrm{Z}\left(I(F)\right) = \psi(F)$ (see the discussion preceding Lemma 11.9), and thus $f(z) = 0$ for $z \in F$ with $y = \psi(z)$. $\square$

Corollary 11.15. *Suppose $r(Z) \in K(Z)$ has no poles in F. Then the function $z \mapsto r(z) : F \to K^{\mathrm{a}}$ belongs to $\mathcal{O}(F)$.*

Proof. Take relatively prime $f, g \in K[Z]$ such that $r(Z) = f(Z)/g(Z)$. Then $g(Z)$ has no zero in F. Now use that by the remark at the beginning of this subsection and Corollary 11.14(ii), the function $z \mapsto g(z) : F \to K^{\mathrm{a}}$ is a unit of $\mathcal{O}(F)$. $\square$

Corollary 11.16. *$\mathcal{O}(F)$ is a principal ideal domain.*

Proof. This follows from Corollaries 11.11 and 11.15 and the fact that the subring $\{r(Z) \in K(Z) : r(Z)$ has no poles in $F\}$ of $K(Z)$ is a localization of $K[Z]$, and thus a principal ideal domain. $\square$

Corollary 11.17. *Let $f_1, f_2 \in \mathcal{O}(F)$ be such that $|f_1(z)| \leqslant |f_2(z)|$ for all $z \in F$. Then f_2 divides f_1 in $\mathcal{O}(F)$.*

Proof. Let $E_1, E_2 \in R\langle Y\rangle$ and $r_1, r_2 \in K(Z)$ be as in Corollary 11.11 for f_1, f_2, respectively. The case where $r_1 = 0$ or $r_2 = 0$ is trivial. Assume $r_1, r_2 \neq 0$. Since E_1, E_2 are units of $R\langle Y\rangle$ we have $|\psi^*(E_1)(z)| = |\psi^*(E_2)(z)| = 1$ for all $z \in F$. Hence $|r_1(z)| \leqslant |r_2(z)|$ for all $z \in F$. Clearly, if $z \in F$ is a zero of r_2 of order ℓ, then z is also a zero of r_1 of order at least ℓ. So $r_1/r_2 \in K(Z)$ has no poles in F, and thus the function $z \mapsto r_2(z)$ in $\mathcal{O}(F)$ divides (in $\mathcal{O}(F)$) the function $z \mapsto r_1(z)$ in $\mathcal{O}(F)$. Therefore f_2 divides f_1 in $\mathcal{O}(F)$. $\square$

The following important result requires more work.

Proposition 11.18. *Suppose $f \in \mathcal{O}(F)$ and $|f(z)| \leqslant 1$ for all $z \in F$. Then $f = \psi^*(g)$ for some $g \in R\langle Y\rangle$.*

Proof. Take $g \in R\langle Y\rangle$ and $c \in R^{\neq}$ such that $f = \psi^*(c^{-1}g)$. Using Lemma 11.9 we can arrange $g = g_1 + \cdots + g_n$ with $g_i \in R\langle Y_i\rangle$ for $i = 1, \ldots, n$, and $g_i(0) = 0$ for $2 \leqslant i \leqslant n$. We can assume $g \neq 0$, so by Lemmas 10.23 and 10.36(i) we can arrange also that one of the g_i has a coefficient outside $\smallfrown(R)$. We prove below that then there is a $z \in F$ such that $|g(\psi(z))| = 1$ (and hence $1 \geqslant |f(z)| = |c^{-1}|$, so $c^{-1}g \in R\langle Y\rangle$, and we are done). Take m, $x \in R^m$, and elements $\sum_{k=0}^{\infty} a_{1k}(X)Y_1^k \in A\langle X, Y_1\rangle$ and $\sum_{k=1}^{\infty} a_{ik}(X)Y_i^k$ in $A\langle X, Y_i\rangle$ for $2 \leqslant i \leqslant n$, such that

$$g_1(Y_1) \;=\; \sum_{k=0}^{\infty} a_{1k}(x)Y_1^k, \qquad g_i \;=\; \sum_{k=1}^{\infty} a_{ik}(x)Y_i^k \;\text{ for }\; i = 2, \ldots, n.$$

Case 1: $|a_{1k}(x)| = 1$ for some k. After rearranging $c_2, \ldots, c_n$ we have $s \in \{1, \ldots, n\}$ such that for $i = 2, \ldots, s$ there is $k \geqslant 1$ with $|a_{ik}(x)| = 1$ and $|\pi_i| = |\pi_1|$, while for $i = s+1, \ldots, n$, either $|a_{ik}(x)| < 1$ for all $k \geqslant 1$, or $|\pi_i| < |\pi_1|$. Suppose $z \in F$ and $|z - c_i| = |\pi_1|$ for $i = 2, \ldots, n$, and set $y := \psi(z)$. If $s < i \leqslant n$ and $|\pi_i| < |\pi_1|$, then $|y_i| < 1$. In any case, if $s < i \leqslant n$, then $|g_i(y_i)| < 1$ by Lemma 10.36(ii). Set $h(Y) := g_1(Y_1) + \cdots + g_s(Y_s) \in R\langle Y_1, \ldots, Y_s\rangle$. By Lemma 10.36(i) we have $k_1, k_2, \ldots, k_s \in \mathbb{N}$ such that $h = h_1 + h_2$ where

$$h_1 \;:=\; \sum_{k=0}^{k_1} a_{1k}(x)Y_1^k + \sum_{i=2}^{s}\sum_{k=1}^{k_i} a_{ik}(x)Y_i^k \in R[Y_1, \ldots, Y_s]$$

and $h_2 \in R\langle Y_1, \ldots Y_s\rangle$ has all its coefficients in $\mathcal{o}(R)$. By Lemma 10.36(ii) we have $|g(y)| = |h(y)| = |h_1(y)| = 1$ for all $y \in (R^{\mathrm{a}})^n$ with $|h_1(y)| = 1$. Using the new indeterminate U we now introduce the rational function $r(U) \in K(U)$ by

$$r(U) \;:=\; h_1\big(U, \frac{\pi_2}{\pi_1 U + c_1 - c_2}, \ldots, \frac{\pi_s}{\pi_1 U + c_1 - c_s}\big).$$

Setting $c_{ik} := (\pi_i/\pi_1)^k$ and $d_i := (c_i - c_1)/\pi_1 \in R$ for $2 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant k_i$ we have $|c_{ik}| = 1$ for those i, k, and $|d_i - d_j| = 1$ for $2 \leqslant i < j \leqslant s$, and

$$r(U) \;=\; \sum_{k=0}^{k_1} a_{1k}(x)U^k + \sum_{i=2}^{s}\sum_{k=1}^{k_i} a_{ik}(x)\frac{c_{ik}}{(U - d_i)^k}.$$

Passing to the image $\overline{r}(U)$ of $r(U)$ in $\boldsymbol{k}(U)$, each of the s resulting summands is nonzero with distinct $\overline{d_2}, \ldots, \overline{d_s} \in \boldsymbol{k}$, so $\overline{r}(U) \neq 0$. This yields infinitely many $u \in R^{\mathrm{a}}$ with $\overline{u} \neq \overline{d}_2, \ldots, \overline{d}_s$, $\overline{u} + \overline{u}_i \neq 0$ for $u_i := (c_1 - c_i)/\pi_1$ with $s < i \leqslant n$, and $|r(u)| = 1$. For such u we take $z := \pi_1 u + c_1$. Then $|z - c_1| \leqslant |\pi_1|$ and $|z - c_i| = |\pi_1|$ for $i = 2, \ldots, n$, so $z \in F$ and $y = \psi(z)$ gives $h_1(y) = r(u)$, and thus $|g(y)| = 1$.

Case 2: $|a_{1k}(x)| < 1$ for all k. One of the g_i has a coefficient outside $\mathcal{o}(R)$, so after rearranging $c_2, \ldots, c_n$ we have $|a_{2k}(x)| = 1$ for some $k \geqslant 1$, and $|\pi_2| \geqslant |\pi_i|$ whenever $2 < i \leqslant n$ and $|a_{ik}(x)| = 1$ for some $k \geqslant 1$. Since $|c_2 - c_i| \geqslant |\pi_2|$ for $2 < i \leqslant n$, we can also arrange $s \in \{2, \ldots, n\}$ is such that for all i with $2 < i \leqslant s$ we have $|a_{ik}(x)| = 1$ for some $k \geqslant 1$ and $|c_2 - c_i| = |\pi_i| = |\pi_2|$, and for all i with $s < i \leqslant n$, either $|a_{ik}(x)| < 1$ for all $k \geqslant 1$, or $|c_2 - c_i| > |\pi_2| \geqslant |\pi_i|$, or $|c_2 - c_i| = |\pi_2| > |\pi_i|$.

Suppose $z \in K$, $|z - c_2| = |\pi_2|$, and $|z - c_i| = |\pi_2|$ whenever $2 < i \leqslant n$ and $|c_2 - c_i| = |\pi_2|$. Then $z \in F$. Set $y := \psi(z)$, so $|g_i(y_i)| < 1$ for $s < i \leqslant n$. Set

$$h(Y) \;:=\; g_2(Y_2) + \cdots + g_s(Y_s) \in R\langle Y_2, \ldots, Y_s\rangle.$$

Lemma 10.36(i) gives $k_2, \ldots, k_s \in \mathbb{N}^{\geqslant 1}$ such that $h = h_1 + h_2$ where

$$h_1 \;:=\; \sum_{i=2}^{s}\sum_{k=1}^{k_i} a_{ik}(x)Y_i^k \in R[Y_2, \ldots, Y_s]$$

and $h_2 \in R\langle Y_2, \ldots Y_s\rangle$ has all its coefficients in $\mathcal{o}(R)$. By Lemma 10.36(ii) we have $|g(y)| = |h(y)| = |h_1(y)| = 1$

for all $y \in (R^{\mathrm{a}})^n$ with $|h_1(y)| = 1$. We now introduce the rational function $r(U) \in K(U)$ by

$$r(U) \ := \ h_1\big(\frac{\pi_2}{\pi_2 U + c_2 - c_2}, \dots, \frac{\pi_s}{\pi_2 U + c_2 - c_s}\big).$$

Setting $c_{ik} := (\pi_i/\pi_2)^k$, $d_i := (c_i - c_2)/\pi_2$ for $2 < i \leqslant s,\ 1 \leqslant k \leqslant k_i$ we have $|c_{ik}| = 1$ for those i, k, $|d_i| = 1$ for $2 < i \leqslant s$, $|d_i - d_j| = 1$ for $2 < i < j \leqslant s$, and

$$r(U) \ = \ \sum_{k=1}^{k_2} a_{2k}(x)\frac{1}{U^k} + \sum_{i=3}^{s}\sum_{k=1}^{k_i} a_{ik}(x)\frac{c_{ik}}{(U-d_i)^k}.$$

As in Case 1 we obtain $u \in R^{\mathrm{a}}$ such that $\overline{u} \neq 0, \overline{d}_3, \dots, \overline{d}_s$, $|r(u)| = 1$, and $\overline{u} + \overline{u}_i \neq 0$ for $u_i := (c_2 - c_i)/\pi_2$ whenever $s < i \leqslant n$ and $|c_2 - c_i| = |\pi_2|$. For such u we take $z := \pi_2 u + c_2$. Then $|z - c_2| = |\pi_2|$, and $|z - c_i| = |\pi_2|$ whenever $2 < i \leqslant n$ and $|c_2 - c_i| = |\pi_2|$, so $z \in F$ and $y = \psi(z)$ gives $h_1(y) = r(u)$, and thus $|g(y)| = 1$. ☐

Corollary 11.19. *$I(F) = \ker(\psi^*)$, and so $\mathcal{O}(F) \cong K\langle Y\rangle/I(F)$ as K-algebras.*

Proof. Let $g \in \ker(\psi^*)$. To show $g \in I(F)$ we can arrange by Lemma 11.9 that $g = g_1 + \cdots + g_n$ with $g_j = g_j(Y_j) \in K\langle Y_j\rangle$ for $j = 1, \dots, n$ and $g_j(0) = 0$ for $j = 2, \dots, n$. We can assume $g \neq 0$, and then the proof of Proposition 11.18 shows that $g(\psi(z)) \neq 0$ for some $z \in F$, contradicting the assumption on g. ☐

Corollary 11.20. *Let $f_1, \dots, f_m \in \mathcal{O}(F)$ be such that $|f_1(z)|, \dots, |f_m(z)| \leqslant 1$ for all $z \in F$, and let $G \in A\langle X\rangle$, $X = (X_1, \dots, X_m)$. Then the function*

$$z \mapsto G\big(f_1(z), \dots, f_m(z)\big) \ : \ F \to R^{\mathrm{a}}$$

belongs to $\mathcal{O}(F)$.

Proof. By Proposition 11.18 we have $f_i = \psi^*(g_i)$ with $g_i \in R\langle Y\rangle$ for $i = 1, \dots, m$, and then for $g := G(g_1, \dots, g_m) \in R\langle Y\rangle$ we have $G\big(f_1(z), \dots, f_m(z)\big) = g\big(\psi(z)\big)$ for all $z \in F$. ☐

Relating $\mathcal{O}(F)$ and $\mathcal{O}_L(F)$. Let L be intermediate between K and K^{a} and of finite degree over K, that is, L is an A-extension of K and a substructure of the $\mathcal{L}^A_{\preccurlyeq}$-structure K^{a}, and $[L : K] < \infty$. Let R_L be the valuation A-ring of L. Then R_L is viable by Lemma 10.26. Thus we have the notion of a connected R_L-affinoid G, and for such G we let $\mathcal{O}_L(G)$ be the corresponding ring of affinoid functions $G \to K^{\mathrm{a}}$. If F is an R-affinoid, then F is also an R_L-affinoid, and $\mathcal{O}(F) = \mathcal{O}_K(F)$ is a subring of $\mathcal{O}_L(F)$.

Lemma 11.21. *Let F be a connected R-affinoid and $b_1, \dots, b_m$ a basis of the vector space L over K. Then the $\mathcal{O}(F)$-module $\mathcal{O}_L(F)$ is free with basis $b_1, \dots, b_m$.*

Proof. Recall that $\mathcal{O}(F) = \psi^*(K\langle Y\rangle) \subseteq \mathcal{R}(F)$. Likewise, $\mathcal{O}_L(F) = \psi_L^*(L\langle Y\rangle) \subseteq \mathcal{R}_L(F)$, where we use the subscript L to indicate the dependence on L. For $g \in L\langle Y\rangle$ we have $g = b_1 f_1 + \cdots + b_m f_m$ with $f_1, \dots, f_m \in K\langle Y\rangle$ by Lemma 10.24, hence

$$\psi_L^*(g) \ = \ b_1\psi^*(f_1) + \cdots + b_m\psi^*(f_m) \in b_1\mathcal{O}(F) + \cdots + b_m\mathcal{O}(F).$$

Thus the $\mathcal{O}(F)$-module $\mathcal{O}_L(F)$ is generated by $b_1, \dots, b_m$.

Suppose $f_1 b_1 + \cdots + f_m b_m = 0$ with $f_1, \dots, f_m \in \mathcal{O}(F)$; it remains to show that then $f_1 = \cdots = f_m = 0$. Consider first the case that the residue field of K has more elements than the number of holes of F. Then Lemma 11.7 gives an infinite set $E \subseteq F \cap R$. If $f_i \neq 0$, then we can take $z \in E$ such that $f_i(z) \neq 0$, but this would contradict $f_1(z)b_1 + \cdots + f_m(z)b_m = 0$ and $b_1, \dots, b_m$ being a basis of L over K. This argument shows $f_1 = \cdots = f_m = 0$. Consider next the case that the residue field $\boldsymbol{k}$ of K is finite with no more elements than the number of holes of F. Then we consider $\boldsymbol{k}$ and the (finite) residue field $\boldsymbol{k}_L$ of L as subfields of the residue field $\boldsymbol{k}^{\mathrm{a}}$ of K^{a}. Take a finite subfield $\boldsymbol{k}_1$ of $\boldsymbol{k}^{\mathrm{a}}$ with $\boldsymbol{k} \subseteq \boldsymbol{k}_1$ such that $\boldsymbol{k}_1$ has more elements than the number of holes of F and $\boldsymbol{k}_1$ is linearly disjoint from $\boldsymbol{k}_L$ over $\boldsymbol{k}$. Let K_1 be a valued subfield of K^{a} such that K_1 is an unramified algebraic extension of K with residue field $\boldsymbol{k}_1$, so $[K_1 : K] = [\boldsymbol{k}_1 : \boldsymbol{k}]$. Then the valued subfield $L_1 := LK_1$ of K^{a} has the property that $[L_1 : L] \leqslant [K_1 : K]$, but also $[L_1 : L] \geqslant [\boldsymbol{k}_L \boldsymbol{k}_1 : \boldsymbol{k}_L] = [\boldsymbol{k}_1 : \boldsymbol{k}]$, hence $[L_1 : L] = [K_1 : K]$, so L and K_1 are linearly disjoint over K. Thus $b_1, \dots, b_m$ is also a basis of the K_1-linear space L_1. Now the earlier argument with K_1 and L_1 instead of K and L show as before that $f_1 = \cdots = f_m = 0$. □

Lemma 11.22. *Let G be a connected R_L-affinoid, F a connected R-affinoid such that $F \subseteq G$, and $g \in \mathcal{O}_L(F)$. Suppose there are infinitely many $z \in F \cap R$ with $g(z) \in K$. Then $g|_F \in \mathcal{O}(F)$.*

Proof. Take a basis $b_1, \dots, b_m$ of the vector space L over K with $b_1 = 1$. Lemma 11.13 gives $g|_F \in \mathcal{O}_L(F)$, so $g|_F = f_1 b_1 + \cdots + f_m b_m$ with $f_1, \dots, f_m \in \mathcal{O}(F)$, by Lemma 11.21. For $z \in F \cap R$ with $g(z) \in K$ we have

$$g(z) \;=\; f_1(z)b_1 + \cdots + f_m(z)b_m \in K,$$

with all $f_i(z) \in K$, so $f_2(z) = \cdots = f_m(z) = 0$, and thus $g(z) = f_1(z)$. Since this is the case for infinitely many $z \in F$ we obtain $g|_F = f_1 \in \mathcal{O}(F)$. □

CHAPTER 12

Analytic AKE-type equivalence and induced structure results

12.1 Introducing Division

To develop our AKE-results for (suitable) valuation A-rings we add "restricted division" as a new primitive. Accordingly we extend the language $\mathcal{L}^A_{\preccurlyeq}$ by a symbol for this division. It will be essential to describe the function given by any one-variable term in this extended language piecewise by affinoid functions. This will force us to pass to A-extensions of finite degree. The key fact about this is Proposition 12.4.

We keep the assumptions from Sections 10.3 and 11.2 on A, R, K, R^{a}, K^{a}. Thus R is viable, K^{a} is an algebraically closed A-extension of K and R^{a} is its valuation A-ring, and so R, K, R^{a}, K^{a} are all $\mathcal{L}^A_{\preccurlyeq}$-structures as specified earlier. For any A-extension L of K with valuation A-ring R_L (in particular for $L = K^{\mathrm{a}}$) we now introduce the restricted division operation $D : L^2 \to L$ by

$$D(a,b) \;:=\; a/b \;\text{ if }\; |a| \leqslant |b| \neq 0, \quad D(a,b) \;:=\; 0 \;\text{ otherwise.}$$

So $D(L^2) \subseteq R_L$. We extend the language $\mathcal{L}^A_{\preccurlyeq}$ to the language $\mathcal{L}^A_{\preccurlyeq,D}$ by adding the binary function symbol D, and expand the $\mathcal{L}^A_{\preccurlyeq}$-structure L accordingly to the $\mathcal{L}^A_{\preccurlyeq,D}$-structure L by interpreting D as above. This makes R, K, and R_L into $\mathcal{L}^A_{\preccurlyeq,D}$-substructures of L. The language $\mathcal{L}^{A,L}_{\preccurlyeq,D}$ is $\mathcal{L}^A_{\preccurlyeq,D}$ augmented by names (constant symbols), one for each element of L, and we construe L accordingly as an $\mathcal{L}^{A,L}_{\preccurlyeq,D}$-structure. Recall the sublanguage $\mathcal{L}_{\preccurlyeq}$ (for valued fields) of $\mathcal{L}^A_{\preccurlyeq}$; augmenting it with names for the elements of L gives the sublanguage $\mathcal{L}^L_{\preccurlyeq}$ of $\mathcal{L}^{A,L}_{\preccurlyeq,D}$.

Describing the A-extension generated over K by an element z. As before, L is an A-extension of K. Let Z be an indeterminate and $z \in L$. Then any $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$ yields an element $\tau(z) \in L$. The set $\{\tau(z) : \tau(Z) \text{ is an } \mathcal{L}^{A,K}_{\preccurlyeq,D}\text{-term}\}$ underlies a substructure of the $\mathcal{L}^A_{\preccurlyeq,D}$-structure L, namely the smallest substructure of the $\mathcal{L}^A_{\preccurlyeq,D}$-structure L that contains $K \cup \{z\}$; we do not claim this set it is the underlying set of a subfield of L. Instead we call attention to the A-closed subring R_z of R_L with underlying set

$$\{\tau(z) : \; \tau(Z) \text{ is an } \mathcal{L}^{A,K}_{\preccurlyeq,D}\text{-term and } \tau(z) \preccurlyeq 1\}.$$

Note: $R \subseteq R_z$; if $z \preccurlyeq 1$, then $z \in R_z$; if $z \succ 1$, then $z^{-1} \in R_z$.

Lemma 12.1. *R_z is a valuation ring dominated by R_L.*

Proof. Let $\tau_1(Z)$, $\tau_2(Z)$ be $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms with $\tau_1(z), \tau_2(z) \preccurlyeq 1$. If $\tau_1(z) \preccurlyeq \tau_2(z)$, then $\tau(Z) := D(\tau_1(Z), \tau_2(Z))$ has the property that $\tau(z) \preccurlyeq 1$ and $\tau_1(z) = \tau(z)\tau_2(z)$, and if $\tau_2(z) \preccurlyeq \tau_1(z)$, then $\tau(Z) := D(\tau_2(Z), \tau_1(Z))$ has the property that $\tau(z) \preccurlyeq 1$ and $\tau_2(z) = \tau(z)\tau_1(z)$. This gives that R_z is a valuation A-ring. $\square$

Let K_z be the fraction field of R_z inside L, equipped with the valuation A-ring R_z. This makes K_z into an A-extension of K, and an $\mathcal{L}^{A}_{\preccurlyeq,D}$-substructure of L. Thus $\tau(z) \in K_z$ for every $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$.

Corollary 12.2. *K_z is the smallest substructure of the $\mathcal{L}^{A}_{\preccurlyeq,D}$-structure L that contains $K \cup \{z\}$ and whose underlying ring is a field. As a consequence, if $z \preccurlyeq 1$, then R_z is the smallest A-closed subring of R_L that contains $R \cup \{z\}$ and whose underlying ring is a valuation ring dominated by R_L.*

Corollary 12.3. *If z is algebraic over K, then $K(z)$ is the underlying field of K_z.*

Proof. Suppose z is algebraic over K. Then the valued subfield $K(z)$ of L expands uniquely to an A-extension of K by Lemma 10.25. This A-extension is then an $\mathcal{L}^{A}_{\preccurlyeq,D}$-substructure of L by Corollary 10.7. Now use Corollary 12.2. $\square$

For our AKE-theory for valuation A-rings we need to understand better what data about z determine the isomorphism type of the A-extension K_z over K. Section 10.4 basically settles this issue for the case when K_z is an immediate A-extension. (In this connection we note that R_z and K_z as defined here agree with R_z and K_z defined in Section 10.4 in the special case considered there.) For the general case we exploit below our results on affinoids.

Towards this goal we now focus on the case $L = K^{\mathrm{a}}$. Given a connected R-affinoid F we can represent any function $f \in \mathcal{O}(F)$ by an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term: let F be given by $(c_1, \dots, c_n; \pi_1, \dots, \pi_n)$, with corresponding map $\psi : F \to (R^{\mathrm{a}})^n$, and let $f = \psi^*(g)$ with $g \in K\langle Y\rangle$, $Y = (Y_1, \dots, Y_n)$. We have $c \in K^\times$, a point $x \in R^m$, and a $G \in A\langle X, Y\rangle$, $X = (X_1, \dots, X_m)$, such that $g = g(Y) = c \cdot G(x, Y)$, and thus $g(y) = c \cdot G(x, y)$ for all $y \in (R^{\mathrm{a}})^n$. Then for all $z \in F$,

$$f(z) \;=\; g\big(\psi(z)\big) \;=\; c \cdot G\big(x, D(z - c_1, \pi_1), D(\pi_2, z - c_2), \dots, D(\pi_n, z - c_n)\big).$$

Conversely, we shall describe the function on R^{a} given by any $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term explicitly as piecewise affinoid, but the domains of the relevant affinoid functions are now connected R_L-affinoids for some A-extension $L \subseteq K^{\mathrm{a}}$ of K with $[L : K] < \infty$. (In [30, Proposition 4.1] a related piecewise description was claimed without allowing a proper finite degree extension, but the proof is defective. The introduction of $\widetilde{A}$ and the generalities underlying it were motivated by having to correct this error.)

Univariate functions given by terms involving restricted division.. In this subsection we let L range over the A-extensions of K that are substructures of the A-extension K^{a} of K and of finite degree over K. For a connected R_L-affinoid F we have the corresponding ring $\mathcal{O}_L(F)$ of affinoid functions $F \to K^{\mathrm{a}}$.

Proposition 12.4. *Let Z be an indeterminate, and $\tau(Z)$ an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. Then there is an L, quantifier-free $\mathcal{L}^{L}_{\preccurlyeq}$-formulas $\phi_1(Z), \dots, \phi_n(Z)$, connected R_L-affinoids $F_1, \dots, F_n$ and functions $f_1 \in \mathcal{O}_L(F_1), \dots, f_n \in \mathcal{O}_L(F_n)$ such that:*

(i) $R^{\mathrm{a}} = \phi_1(R^{\mathrm{a}}) \cup \dots \cup \phi_n(R^{\mathrm{a}})$;

(ii) $\phi_j(R^{\mathrm{a}}) \subseteq F_j$ *and* $\tau(z) = f_j(z)$ *for all* $z \in \phi_j(R^{\mathrm{a}})$, *for* $j = 1, \dots, n$.

Proof. By induction on the complexity of $\tau = \tau(Z)$. For τ the name of an element of K or just the variable Z one can take $L = K$, $n = 1$, and make the obvious choices of ϕ_1, F_1, f_1. Next, given $L, \phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$ as in the proposition, we only need to replace each f_j by $-f_j$ to make it work for $-\tau$ instead of τ.

Suppose $\tau = \tau_1 + \tau_2$. The inductive assumption gives L, quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formulas $\phi_{11}(Z), \dots, \phi_{1n_1}(Z)$ and $\phi_{21}(Z), \dots, \phi_{2n_2}(Z)$, connected R_L-affinoids

$$F_{11}, \dots, F_{1n_1}, F_{21}, \dots, F_{2n_2},$$

and $f_{ij} \in \mathcal{O}_L(F_{ij})$ for $i = 1, 2$ and $j = 1, \dots, n_i$ such that

- $R^{\mathrm{a}} = \phi_{11}(R^{\mathrm{a}}) \cup \cdots \cup \phi_{1n_1}(R^{\mathrm{a}}) = \phi_{21}(R^{\mathrm{a}}) \cup \cdots \cup \phi_{2n_2}(R^{\mathrm{a}})$;
- $\phi_{ij}(R^{\mathrm{a}}) \subseteq F_{ij}$ and $\tau_i(z) = f_{ij}(z)$ for all $z \in \phi_{ij}(R^{\mathrm{a}})$.

Let $1 \leqslant j_1 \leqslant n_1$ and $1 \leqslant j_2 \leqslant n_2$ and set $\phi_{j_1 j_2} := \phi_{1j_1} \wedge \phi_{2j_2}$ and $F_{j_1 j_2} := F_{1j_1} \cap F_{2j_2}$. Then $\phi_{j_1 j_2}(R^{\mathrm{a}}) \subseteq F_{j_1 j_2}$ and $\tau(z) = f_{1j_1}(z) + f_{2j_2}(z)$ for $z \in \phi_{j_1 j_2}(R^{\mathrm{a}})$. Thus listing the nonempty $F_{j_1 j_2}$ as $F_1, \dots, F_n$, the corresponding $f_{1j_1}|_{F_{j_1 j_2}} + f_{2j_2}|_{F_{j_1 j_2}}$ as $f_1, \dots, f_n$, and the corresponding $\phi_{j_1 j_2}$ as $\phi_1, \dots, \phi_n$ yields (i) and (ii). This also uses Lemma 11.13. The case $\tau = \tau_1 \cdot \tau_2$, is handled in the same way.

Next, suppose $\tau = D(\tau_1, \tau_2)$, and let the $\phi_{ij}, F_{ij}, f_{ij}$ be as before and also define $\phi_{j_1 j_2}$ and $F_{j_1 j_2}$ as before. Consider one such pair $j = (j_1, j_2)$ and set $\phi_j = \phi_{j_1 j_2}$ and $F_j = F_{j_1 j_2}$. By increasing L and using Corollaries 11.11 and 11.6 we obtain

$$\{z \in F_j : \ |f_{1j_1}(z)| \leqslant |f_{2j_2}(z)|\} \ = \ F^{j,1} \cup \cdots \cup F^{j,n} \cup E$$

where $F^{j,1}, \dots, F^{j,n}$ are connected R_L-affinoids contained in F_j and $E \subseteq F_j \cap R_L$ is finite. Let $1 \leqslant \nu \leqslant n$ and take $f^{j,\nu} \in \mathcal{O}_L(F^{j,\nu})$ with $f_{1j_1}(z) = f^{j,\nu}(z) f_{2j_2}(z)$ for all $z \in F^{j,\nu}$; such an $f^{j,\nu}$ exists by Lemma 11.13 and Corollary 11.17. Corollary 11.11 gives a quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formula $\phi^{j,\nu}(Z)$ such that for all $z \in R^{\mathrm{a}}$,

$$R^{\mathrm{a}} \models \phi^{j,\nu}(z) \iff z \in F^{j,\nu} \text{ and } f_{2j_2}(z) \neq 0.$$

Thus $\phi^{j,\nu}(R^{\mathrm{a}}) \subseteq F^{j,\nu}$, and for $z \in (\phi_j \wedge \phi^{j,\nu})(R^{\mathrm{a}})$ we have $\tau(z) = f^{j,\nu}(z)$. For $a \in E$ we have $b := \tau(a) \in R_L$, $(Z = a)(R^{\mathrm{a}}) = \{a\} \subseteq R^{\mathrm{a}}$, and so for all $z \in (Z = a)(R^{\mathrm{a}})$ we have $\tau(z) = b$. Corollary 11.11 also gives a quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formula $\theta_j(Z)$ such that for all $z \in R^{\mathrm{a}}$,

$$R^{\mathrm{a}} \models \theta_j(z) \iff z \in F_j \text{ and } \big(|f_{1j_1}(z)| > |f_{2j_2}(z)| \text{ or } f_{2j_2}(z) = 0\big).$$

Thus $\theta_j(R^{\mathrm{a}}) \subseteq F_j$, and for $z \in (\phi_j \wedge \theta_j)(R^{\mathrm{a}})$ we have $\tau(z) = 0$. Moreover, for the various pairs j above we can take the same increased L.

Finally, suppose $\tau = G(\tau_1, \dots, \tau_m)$. The inductive assumption gives L and for $i = 1, \dots, m$ quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formulas $\phi_{i1}(Z), \dots, \phi_{in_i}(Z)$, connected R_L-affinoids $F_{i1}, \dots, F_{in_i}$, and functions $f_{i1} \in \mathcal{O}_L(F_{i1}), \dots, f_{in_i} \in \mathcal{O}_L(F_{in_i})$ such that

- $R^{\mathrm{a}} = \phi_{i1}(R^{\mathrm{a}}) \cup \cdots \cup \phi_{in_i}(R^{\mathrm{a}})$;
- $\phi_{ij}(R^{\mathrm{a}}) \subseteq F_{ij}$ and $\tau_i(z) = f_{ij}(z)$ for all $z \in \phi_{ij}(R^{\mathrm{a}})$ and $j = 1, \dots, n_i$.

Let $j = (j_1, \dots, j_m)$ with $1 \leqslant j_1 \leqslant n_1, \dots, 1 \leqslant j_m \leqslant n_m$ and set

$$\phi_j := \phi_{1j_1} \wedge \cdots \wedge \phi_{mj_m}, \quad F_j := F_{1j_1} \cap \cdots \cap F_{mj_m}.$$

By increasing L and using Corollories 11.11 and 11.5 we arrange that

$$\{z \in F_j : \ |f_{1j_1}(z)| \leqslant 1, \dots, |f_{mj_m}(z)| \leqslant 1\} \ = \ F^{j,1} \cup \cdots \cup F^{j,n}$$

where $F^{j,1}, \dots, F^{j,n}$ are connected R_L-affinoids. Let $1 \leqslant \nu \leqslant n$ and take a quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formula $\phi^{j,\nu}(Z)$ such that for all $z \in R^{\mathrm{a}}$,

$$R^{\mathrm{a}} \models \phi^{j,\nu}(z) \iff z \in F^{j,\nu},$$

and for $i = 1, \dots, m$, set $f^{j,\nu}_{ij_i} := f_{ij_i}|_{F^{j,\nu}} \in \mathcal{O}_L(F^{j,\nu})$, so by Corollary 11.20,

$$f^{j,\nu} \ := \ G\big(f^{j,\nu}_{1j_1}, \dots, f^{j,\nu}_{mj_m}\big) \in \mathcal{O}_L(F^{j,\nu}).$$

Then $\phi^{j,\nu}(R^{\mathrm{a}}) = F^{j,\nu}$ and $\tau(z) = f^{j,\nu}(z)$ for all $z \in (\phi_j \wedge \phi^{j,\nu})(R^{\mathrm{a}})$. Also $\tau(z) = 0$ for all $z \in (\phi_j \wedge \neg\phi^{j,1} \wedge \neg\phi^{j,2} \wedge \cdots \wedge \neg\phi^{j,n})(R^{\mathrm{a}}) \subseteq R^{\mathrm{a}}$. ☐

Corollary 12.5. *For $z \in R^{\mathrm{a}}$ and K_{alg} the algebraic closure of K in K^{a},*

$$\operatorname{res} K_z \ \subseteq \ \operatorname{res} K_{\mathrm{alg}}(z) \ \subseteq \ \operatorname{res} K^{\mathrm{a}}, \qquad v(K_z^{\times}) \ \subseteq \ v\big(K_{\mathrm{alg}}(z)^{\times}\big) \subseteq \ v\big((K^{\mathrm{a}})^{\times}\big).$$

As a consequence, $\Gamma = v(K^{\times})$ and $v(K_z^{\times})$ have the same cardinality, and if $\operatorname{res} K$ is infinite, then $\operatorname{res} K$ and $\operatorname{res} K_z$ have the same cardinality.

Proof. Consider a nonzero element $\tau(z)$ of K_z, where $\tau(Z)$ is an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. Let L and $\phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$ be as in Proposition 12.4. Take $j \in \{1, \dots, n\}$ with $z \in \phi_j(R^{\mathrm{a}})$. Then Corollary 11.11 applied to $L,\ F_j,\ f_j$ in the role of $K,\ F,\ f$ yields $\rho(Z) \in L(Z)$ without poles in F such that $\tau(z) \sim \rho(z)$. This gives the desired inclusions. The rest now follows from [29, Corollary 5.19]. ☐

The recursive construction of formulas, affinoids, and functions in the proof of Proposition 12.4 gives further information recorded below. First we define for $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms τ of the form $\tau(Z)$ its complexity $c(\tau)$ by recursion: $c(\tau) := 0$ for τ the name of an element of K or the variable Z, $c(\tau) := 1 + \sum_{i=1}^{m} c(\tau_i)$ if τ is $H(\tau_1, \dots, \tau_m)$ for a function symbol H in $\mathcal{L}^{A}_{\preccurlyeq,D}$ of arity m and $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms $\tau_i(Z)$.

Corollary 12.6. *The recursive construction in the proof of Proposition 12.4 yields $\phi_1, \dots, \phi_n,\ F_1, \dots, F_n$ such that for all $j \in \{1, \dots, n\}$ one of the following holds:*

- *$\phi_j(R^{\mathrm{a}})$ is finite;*
- *$\tau(z) = \tau'(z)$ for an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau'(Z)$ with $c(\tau') < c(\tau)$ and all $z \in \phi_j(R^{\mathrm{a}})$;*
- *$\phi_j(R^{\mathrm{a}}) = F_j \setminus E_j$ with finite $E_j \subseteq F_j$.*

The proof of Corollary 12.6 is by a routine induction on $c(\tau)$ and close inspection of the constructions in the proof of Proposition 12.4.

Uniformity with respect to K^{a}. To enable model-theoretic arguments we need

$$\phi_1, \ldots, \phi_n, F_1, \ldots, F_n, f_1, \ldots, f_n$$

in Proposition 12.4 to be in some sense independent of K^{a}. To state this accurately we introduce an algebraic closure K_{alg} of K. By Lemma 10.25, K_{alg} expands uniquely to an A-extension of K. We let K_{alg} also denote this A-extension, with R_{alg} as its A-valuation ring. In the rest of this section Z is an indeterminate.

Let K^{a} be any algebraically closed A-extension of K. Then there exists a field embedding $K_{\mathrm{alg}} \to K^{\mathrm{a}}$ over K. Let $\imath$ be such a field embedding. Then $\imath$ is also an $\mathcal{L}_{\preccurlyeq}$-embedding by [2, Proposition 3.3.11]; alternatively, use [29, Corollary 3.16] to get such a map. The $\mathcal{L}_{\preccurlyeq}$-theory of algebraically closed valued fields with nontrivial valuation has quantifier elimination [29, Theorem 3.29], so

$$(\star) \qquad \imath(K_{\mathrm{alg}}) \text{ is an elementary } \mathcal{L}_{\preccurlyeq}\text{-substructure of } K^{\mathrm{a}}.$$

For any $\mathcal{L}_{\preccurlyeq,D}^{K_{\mathrm{alg}}}$-formula $\phi(Z)$, let ${}^{\imath}\phi(Z)$ be the $\mathcal{L}_{\preccurlyeq,D}^{K^{\mathrm{a}}}$-formula obtained by replacing every occurrence of a name of an element $a \in K_{\mathrm{alg}}$ in ϕ by the name of $\imath(a) \in K^{\mathrm{a}}$. For $\mathcal{L}_{\preccurlyeq,D}^{K_{\mathrm{alg}}}$-definable $P \subseteq K_{\mathrm{alg}}$, let ${}^{\imath}P$ be the corresponding definable subset of K^{a}: if the $\mathcal{L}_{\preccurlyeq,D}^{K_{\mathrm{alg}}}$-formula $\phi(Z)$ defines P in K_{alg}, then ${}^{\imath}\phi(Z)$ defines ${}^{\imath}P$ in K^{a}. Thus if P is a connected R_{alg}-affinoid in the sense of the nontrivially valued field K_{alg}, then ${}^{\imath}P$ is a connected R^{a}-affinoid in the sense of the nontrivially valued field K^{a}.

By Corollary 10.8, $\imath$ is also an $\mathcal{L}_{\preccurlyeq,D}^{A}$-embedding, and we view $\imath(K_{\mathrm{alg}})$ accordingly as an $\mathcal{L}_{\preccurlyeq,D}^{A}$-substructure of K^{a}. Let L range over the A-extensions of K such that $[L:K] < \infty$ and L is a substructure of the A-extension K_{alg} of K. Let F be a connected R_L-affinoid and consider the K-algebra $\mathcal{O}_L(F)$, both in the sense of K_{alg} as the ambient A-extension of K. This yields the connected $R_{\imath(L)}$-affinoid ${}^{\imath}F$ and the K-algebra $\mathcal{O}_{\imath(L)}({}^{\imath}F)$, both in the sense of the A-extension K^{a} of K.

Lemma 12.7. *For each $f \in \mathcal{O}_L(F)$ there is a unique ${}^{\imath}f \in \mathcal{O}_{\imath(L)}({}^{\imath}F)$ such that $\imath\big(f(z)\big) = {}^{\imath}f\big(\imath(z)\big)$ for all $z \in F$. Moreover, the map*

$$f \mapsto {}^{\imath}f \;:\; \mathcal{O}_L(F) \to \mathcal{O}_{\imath(L)}({}^{\imath}F)$$

is an isomorphism of K-algebras that commutes with restriction: for a connected R_L-affinoid $F' \subseteq F$ and $f \in \mathcal{O}_L(F)$ we have ${}^{\imath}F' \subseteq {}^{\imath}F$ and ${}^{\imath}\big(f|_{F'}\big) = ({}^{\imath}f)|_{{}^{\imath}F'}$.

Proof. Let $F \subseteq R_{\mathrm{alg}}$ be given by $(c_1, \ldots, c_n; \pi_1, \ldots, \pi_n)$, with corresponding map $\psi : F \to (R_{\mathrm{alg}})^n$. Then ${}^{\imath}F \subseteq R^{\mathrm{a}}$ is likewise given by

$$\big(\imath(c_1), \ldots, \imath(c_n); \imath(\pi_1), \ldots, \imath(\pi_n)\big),$$

and we have the corresponding map ${}^{\imath}\psi : {}^{\imath}F \to (R^{\mathrm{a}})^n$. Next, let $f \in \mathcal{O}_L(F)$ and take $g = g(Y) \in L\langle Y \rangle$, $Y = (Y_1, \ldots, Y_n)$ such that $f = \psi^*(g)$. Take $c \in K^{\times}$, a point $x \in R_L^m$, and a $G \in A\langle X, Y \rangle$, $X = (X_1, \ldots, X_m)$, such that $g(Y) = c \cdot G(x, Y)$. Then, with $\imath(x) := \big(\imath(x_1), \ldots, \imath(x_m)\big)$, we set

$${}^{\imath}g(Y) \;:=\; \imath(c) \cdot G\big(\imath(x), Y\big) \in \imath(L)\langle Y \rangle,$$

and ${}^{\imath}f := {}^{\imath}\psi^*\big({}^{\imath}g\big) \in \mathcal{O}_{\imath(L)}({}^{\imath}F)$. It is easy to check that then $\imath\big(f(z)\big) = {}^{\imath}f\big(\imath(z)\big)$ for all $z \in F$. Uniqueness, and the "Moreover" claim follow from Corollary 11.14(i). □

With the above notational conventions we let

$$\big(L; (\phi_j, \Phi_j, F_j, f_j)_{j=1}^n\big)$$

denote a tuple such that for $j = 1, \dots, n$:

1. ϕ_j and Φ_j are quantifier-free $\mathcal{L}^L_{\preccurlyeq}$-formulas $\phi_j(Z)$ and $\Phi_j(Z)$;
2. F_j is a connected R_L-affinoid and $f_j \in \mathcal{O}_L(F_j)$;
3. $\Phi_j(K_{\text{alg}}) = F_j$.

Here "connected R_L-affinoid" and "$\mathcal{O}_L(F_j)$" are in the sense of K_{alg} as ambient algebraically closed A-extension of K.

Let now $\tau(Z)$ be an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. The tuple $\big(L; (\phi_j, \Phi_j, F_j, f_j)_{j=1}^n\big)$ is said to be *good for τ in $(K^{\text{a}}, \imath)$* if, in addition to (1), (2), (3), the following hold:

(4) $\phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$ satisfy (i) and (ii) in Proposition 12.4 with K_{alg} in the role of K^{a} there;

(5) $\tau(z) = {}^{\imath}f_j(z)$ for all $z \in {}^{\imath}\phi_j(R^{\text{a}})$ and $j = 1, \dots, n$;

Using ($\star$), (4) implies $R^{\text{a}} = {}^{\imath}\phi_1(R^{\text{a}}) \cup \dots \cup {}^{\imath}\phi_n(R^{\text{a}})$ and

$$ {}^{\imath}\phi_1(R^{\text{a}}) \ \subseteq\ {}^{\imath}F_1,\ \dots,\ {}^{\imath}\phi_n(R^{\text{a}}) \ \subseteq\ {}^{\imath}F_n.$$

Thus (5) makes sense if (4) holds.

Corollary 12.8. *There exists a tuple $\big(L; (\phi_j, \Phi_j, F_j, f_j)_{j=1}^n\big)$ such that for every algebraically closed A-extension K^{a} of K and field embedding $\imath : K_{\text{alg}} \to K^{\text{a}}$ over K this tuple is good for τ in $(K^{\text{a}}, \imath)$.*

Proof. The construction of such a tuple is by recursion on τ, as in the proof of Proposition 12.4: Let i be the inclusion $K \to K_{\text{alg}}$. We follow the steps in that proof for (K_{alg}, i) in the role of $(K^{\text{a}}, \imath)$, and observe that these steps then work for any $(K^{\text{a}}, \imath)$, by ($\star$) and Lemma 12.7. $\square$

The quantifier-free type of an element over K. First we fix some terminology and notation. For a valued field E and an element z in a valued field extension F of E, the quantifier-free $\mathcal{L}_{\preccurlyeq}$-type of z over E is the set $\text{qftp}_{\preccurlyeq}(z|E)$ of all quantifier-free $\mathcal{L}^E_{\preccurlyeq}$-formulas $\theta(Z)$ such that $F \models \theta(z)$. Likewise, for an element z in an A-extension F of K, the quantifier-free $\mathcal{L}^A_{\preccurlyeq,D}$-type of z over K is the set $\text{qftp}^A_{\preccurlyeq,D}(z|K)$ of all quantifier-free $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-formulas $\theta(Z)$ such that $F \models \theta(z)$.

The previous subsections are the ingredients for the following key fact:

Proposition 12.9. *Let K_1 and K_2 be A-extensions of K, and suppose $z_1 \in K_1$ and $z_2 \in K_2$ satisfy $\text{qftp}_{\preccurlyeq}(z_1|K) = \text{qftp}_{\preccurlyeq}(z_2|K)$. Then*

$$\text{qftp}^A_{\preccurlyeq,D}(z_1|K) \ = \ \text{qftp}^A_{\preccurlyeq,D}(z_2|K).$$

Proof. Our assumption gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $i : K(z_1) \to K(z_2)$ over K which sends z_1 to z_2. If z_1 is algebraic over K, then so is z_2 and $K(z_1)$ and $K(z_2)$ underly the $\mathcal{L}^A_{\preccurlyeq,D}$-substructures K_{z_1} and K_{z_2} of K_1

and K_2, respectively, by Corollary 12.3, so i is an $\mathcal{L}^A_{\preccurlyeq,D}$-isomorphism over K by Corollary 10.8. Hence z_1 and z_2 have the same quantifier-free $\mathcal{L}^A_{\preccurlyeq,D}$-type over K.

The remaining case is that z_1 and z_2 are both transcendental over K. Replacing z_1, z_2 by their reciprocals if necessary, we arrange $z_1, z_2 \preccurlyeq 1$. We claim that for every $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$,

$$\tau(z_1) = 0 \iff \tau(z_2) = 0.$$

For c, d in any A-extension of K, $c \preccurlyeq d$ if and only if $c = 0$ or $D(c,d) \neq 0$. Hence, in light of Corollary 12.2, our claim yields an $\mathcal{L}^A_{\preccurlyeq,D}$-isomorphism $K_{z_1} \to K_{z_2}$ over K given by $\tau(z_1) \mapsto \tau(z_2)$, where $\tau(Z)$ ranges over $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms. Thus a proof of the claim will complete the proof of the proposition.

By passing to algebraic closures we arrange that K_1 and K_2 are algebraically closed. Let $\tau(Z)$ be an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term such that $\tau(z_1) = 0$. Take a tuple

$$\big(L; (\phi_j, \Phi_j, F_j, f_j)_{j=1}^n\big)$$

as in Corollary 12.8 and field embeddings $\imath : K_{\text{alg}} \to K_1$ and $\jmath : K_{\text{alg}} \to K_2$ over K. This gives $j \in \{1, \dots, n\}$ with $z_1 \in {}^{\imath}\phi_j(R^{\mathrm{a}})$, and thus $z_2 \in {}^{\jmath}\phi_j(R^{\mathrm{b}})$. Then $\tau(z_1) = {}^{\imath}f_j(z_1) = 0$, so $f_j = 0$ by Corollary 11.11 and z_1 being transcendental over K. Hence $\tau(z_2) = {}^{\imath}f_j(z_2) = 0$. This proves the forward direction of our claim. The backward direction follows in the same way. $\square$

When is K_z an immediate extension of $K(z)$?. Let K^{a} be an algebraically closed A-extension of K and $z \in K^{\mathrm{a}}$. It is plausible that K_z is then always an immediate extension of its valued subfield $K(z)$. Indeed, this is the case when z is algebraic over K by Corollary 12.3, and also when $\operatorname{char} \boldsymbol{k} = 0$ and $K(z)$ is an immediate extension of K, by Proposition 10.40 and [27, Corollaries 4.16, 4.22].

In the next section, where $\operatorname{char} \boldsymbol{k} = 0$, we require such immediacy in two more cases, treated in Propositions 12.10 and 12.12 below.

To prepare for the proof of Proposition 12.10 we consider a valued subfield L of K^{a} such that $K \subseteq L$ and $[L : K] = m$. Suppose $z \in R^{\mathrm{a}}$ and $\overline{z}$ is transcendental over $\boldsymbol{k}$. Then [29, Lemma 3.22] gives for the valued subfields $K(z) \subseteq L(z)$ of K^{a} (in addition to $[L(z) : K(z)] = m$):

$$\Gamma_{K(z)} \;=\; \Gamma, \quad \Gamma_{L(z)} = \Gamma_L, \quad \operatorname{res} K(z) \;=\; \boldsymbol{k}(\overline{z}), \quad \operatorname{res} L(z) \;=\; \boldsymbol{k}_L(\overline{z}).$$

In the proofs of the next two propositions L ranges over the A-extensions of K that are substructures of the A-extension K^{a} of K and of finite degree over K.

Proposition 12.10. *Assume that $\boldsymbol{k}$ is infinite. Let $z \in R^{\mathrm{a}}$, and suppose that $\overline{z}$ is transcendental over $\boldsymbol{k}$. Then $v(K_z^{\times}) = \Gamma$ and $\operatorname{res} K_z = \boldsymbol{k}(\overline{z})$, so K_z is an immediate extension of $K(z)$.*

Proof. Consider an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$ such that $\tau(z) \neq 0$. Proposition 12.4 and Corollary 11.11 give L and $\rho(Z) \in L(Z)$ such that z is not a pole of ρ and

$$\tau(z) \sim \rho(z) \in L(z).$$

Hence $\tau(z) \asymp b$ with $b \in L^{\times}$. Then $\tau(Z) \asymp b$ is in $\operatorname{qftp}^A_{\preccurlyeq,D}(z|L)$. Proposition 12.9 with L instead of K says that $\operatorname{qftp}^A_{\preccurlyeq,D}(z|L)$ is determined by its subset $\operatorname{qftp}_{\preccurlyeq}(z|L)$. Moreover, $\operatorname{qftp}_{\preccurlyeq}(z|L)$ contains all $\mathcal{L}^L_{\preccurlyeq}$-formulas $q(Z) \asymp 1$ with monic $q(Z) \in R_L[Z]$, and is determined by these formulas since $\overline{z}$ is transcendental over $\boldsymbol{k}_L$.

Hence (the punch line) compactness and taking products gives a single monic $q(Z) \in R_L[Z]$ such that for all $u \in R^{\mathrm{a}}$, if $q(u) \asymp 1$, then $\tau(u) \asymp b$. Since $\boldsymbol{k}$ is infinite we have $u \in R$ such that $\overline{q}(\overline{u}) \neq 0$. Then $q(u) \asymp 1$, so $\tau(u) \asymp b$, and $\tau(u) \in K$, so $v(\tau(z)) = v(b) = v(\tau(u)) \in \Gamma$. This concludes the proof of $v(K_z^\times) = \Gamma$.

To obtain $\operatorname{res} K_z = \boldsymbol{k}(\overline{z})$ we show for $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms $\tau(Z)$ that

$$\tau(z) \preccurlyeq 1 \quad \Rightarrow \quad \operatorname{res} \tau(z) \in \boldsymbol{k}(\overline{z}).$$

If $\tau(z) \prec 1$, then $\operatorname{res} \tau(z) = 0$ and we are done, so assume $\tau(z) \asymp 1$. Proposition 12.4 and Corollary 11.11 give L and $\rho(Z) \in L(Z)$ such that z is not a pole of ρ and $\tau(z) \sim \rho(z) \in L(z)$. Suppose towards a contradiction that $\operatorname{res} \tau(z) \notin \boldsymbol{k}(\overline{z})$. With $[L:K] = m$, take $b_1, \dots, b_m \in R_L$ with $b_1 = 1$ such that $\overline{b}_1, \dots, \overline{b}_r$ is a basis of $\boldsymbol{k}_L$ over $\boldsymbol{k}$ and thus of $\boldsymbol{k}_L(\overline{z})$ over $\boldsymbol{k}(\overline{z})$. Then $\tau(z) \sim \rho(z) \sim b_1\rho_1(z) + \cdots + b_m\rho_m(z)$ where $\rho_1(Z), \dots, \rho_m(Z) \in K(Z)$, $\rho_i(z) \preccurlyeq 1$ for $i = 1, \dots, m$, and $\rho_i(z) \asymp 1$ for some $i \in \{2, \dots, m\}$, say for $i = 2$. Hence

$$\tau(Z) \sim b_1\rho_1(Z) + \cdots + b_m\rho_m(Z) \wedge \rho_1(Z) \preccurlyeq 1 \wedge \cdots \wedge \rho_m(Z) \preccurlyeq 1 \wedge \rho_2(Z) \asymp 1$$

belongs to $\operatorname{qftp}^A_{\preccurlyeq,D}(z|L)$. Using Proposition 12.9 and compactness as before yields a monic $q(Z) \in R_L[Z]$ such that for all $u \in R^{\mathrm{a}}$, if $q(u) \asymp 1$, then

$$\tau(u) \sim b_1\rho_1(u) + \cdots + b_m\rho_m(u), \quad \rho_1(u), \dots, \rho_m(u) \preccurlyeq 1, \quad \rho_2(u) \asymp 1.$$

Now take $u \in R$ such tht $q(u) \asymp 1$. Then $\tau(u) \in K$, a contradiction. $\square$

To prepare for the next proposition, recall from [29, Lemma 3.23] that if $z \neq 0$ and $dv(z) \notin \Gamma$ for all $d \geqslant 1$, then with $[L:K] = m$ we obtain $[L(z):K(z)] = m$, and

$$\Gamma_{K(z)} \ = \ \Gamma + \mathbb{Z}v(z), \quad \Gamma_{L(z)} = \ \Gamma_L + \mathbb{Z}v(z), \quad \operatorname{res} K(z) \ = \ \boldsymbol{k}, \quad \operatorname{res} L(z) \ = \ \boldsymbol{k}_L.$$

Instead of requiring that $\boldsymbol{k}$ be infinite we shall impose in the next proposition that Γ is a $\mathbb{Z}$-group and that R_z is viable. First a lemma:

Lemma 12.11. *Assume Γ is a $\mathbb{Z}$-group, R_z is viable, $z \neq 0$, and $dv(z) \notin \Gamma$ for all $d \geqslant 1$. Suppose $z^e \prec b$, where $e \geqslant 1$, $b \in K^\times$. Then $z^e \prec u^e \prec b$ for some $u \in K^\times$.*

Proof. With $\alpha := ev(z)$ and $\beta := vb$ we have $\beta < \alpha$, and thus $\beta + n < \alpha$ for all n, where $n = v(t^n)$ with $t \in R$ such that $\smallint(R) = tR$: otherwise $\beta + n < \alpha < \beta + (n+1)$ for some n, and hence $0 < \alpha - (\beta + n) < 1$ for such n, which by Corollary 10.28 yields a $d \geqslant 1$ with $d \cdot \big(\alpha - (\beta + n)\big) \in \Gamma$, and thus $dev(z) \in \Gamma$ contradicting the assumption on z. Since Γ is a $\mathbb{Z}$-group we can take $n \geqslant 1$ such that $\beta + n = e\gamma$ with $\gamma \in \Gamma$. Then $u \in K^\times$ with $v(u) = \gamma$ gives $z^e \prec u^e \prec b$. $\square$

Proposition 12.12. *Assume Γ is a $\mathbb{Z}$-group, R_z is viable, $z \neq 0$, and $dv(z) \notin \Gamma$ for all $d \geqslant 1$. Then K_z is an immediate extension of $K(z)$, equivalently,*

$$v(K_z^\times) \ = \ \Gamma + \mathbb{Z}v(z), \qquad \operatorname{res}(K_z) \ = \ \boldsymbol{k}.$$

Proof. To emulate the idea behind the proof of Proposition 12.10 we describe $\operatorname{qftp}_{\preccurlyeq}(z|K)$. Let b, b_1, b_2 range over $K^\times$ and e over $\mathbb{N}^{\geqslant 1}$. If $vz > \Gamma$, then $\operatorname{qftp}_{\preccurlyeq}(z|K)$ contains all formulas $0 \neq Z \prec b$ and is by [29, Lemma 3.23] determined by those formulas, so by Proposition 12.9 and compactness there is for any formula $\theta^A(Z)$

in $\operatorname{qftp}^{A}_{\preccurlyeq,D}(z|K)$ a b such that $K^{\mathrm{a}} \models 0 \neq Z \prec b \to \theta^A(Z)$. Likewise, if $vz < \Gamma$, there is for any formula $\theta^A(Z)$ in $\operatorname{qftp}^{A}_{\preccurlyeq,D}(z|K)$ a b such that $K^{\mathrm{a}} \models b \prec Z \to \theta^A(Z)$. The remaining case is that $\gamma_1 < vz < \gamma_2$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then $\operatorname{qftp}_{\preccurlyeq}(z|K)$ contains all formulas $b_1 \prec Z^e \prec b_2$ such that $b_1 \prec z^e \prec b_2$, and is determined by these formulas in view of [29, Lemma 3.23]. As before we see that for any formula $\theta^A(Z)$ in $\operatorname{qftp}^{A}_{\preccurlyeq,D}(z|K)$ there are b_1, b_2 with $b_1 \prec z^e \prec b_2$ such that $K^{\mathrm{a}} \models b_1 \prec Z^e \prec b_2 \to \theta^A(Z)$. So far we did not use the assumption that Γ is a $\mathbb{Z}$-group and R_z is viable.

To obtain $v(K_z^\times) = \Gamma + \mathbb{Z}v(z)$ we consider an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$ with $\tau(z) \neq 0$; our job is to show that then $v(\tau(z)) \in \Gamma + v(z)\mathbb{Z}$. As in the proof of Proposition 12.10 we have L and $\rho(Z) \in L(Z)$ such that z is not a pole of ρ and $\tau(z) \sim \rho(z)$, so

$$v(\tau(z)) \;=\; v(\rho(z)) \;=\; \gamma + kv(z), \qquad \gamma \in \Gamma_L,\ k \in \mathbb{Z}.$$

Suppose $\gamma \notin \Gamma$. Take $d \geqslant 1$ such that $d\gamma \in \Gamma$, so $d\gamma = vb$ with $vb \notin d\Gamma$. Then $\tau(z)^d \asymp b \cdot z^{kd}$, so $\tau(Z)^d \asymp b \cdot Z^{kd}$ belongs to $\operatorname{qftp}^{A,K}_{\preccurlyeq,D}(z|K)$. At this point we use the assumption that Γ is a $\mathbb{Z}$-group and R_z is viable: Using Lemma 12.11 and the facts about $\operatorname{qftp}_{\preccurlyeq}(z|K)$ stated in the beginning of the proof we obtain $u \in K^\times$ such that $\tau(u)^d \asymp b \cdot u^{kd}$, which in combination with $\tau(u) \in K^\times$ gives $v(b) \in d\Gamma$, a contradiction.

To obtain $\operatorname{res} K_z = \boldsymbol{k}$, let $\tau(Z)$ be as before with $\tau(z) \asymp 1$; our job is to show that $\operatorname{res}\big(\tau(z)\big) \in \boldsymbol{k}$. With L and $\rho(Z)$ as before we have $\operatorname{res}\tau(z) = \operatorname{res}\rho(z) \in \boldsymbol{k}_L$. Take a monic $q(Z) \in R[Z]$ such that $\overline{q}(Z) \in \boldsymbol{k}[Z]$ is irreducible and $q(\tau(z)) \prec 1$. Then $\tau(Z) \asymp 1 \wedge q(\tau(Z)) \prec 1$ is in $\operatorname{qftp}^{A,K}_{\preccurlyeq,D}(z|K)$, and so the usual argument gives $u \in K^\times$ such that $\tau(u) \asymp 1$ and $q(\tau(u)) \prec 1$. But then $\tau(u) \in K$, so the irreducible $\overline{q}(Z) \in \boldsymbol{k}[Z]$ must have degree 1, and so $\operatorname{res}\tau(z) \in \boldsymbol{k}$. $\square$

12.2 An analytic Equivalence Theorem

We begin with some terminology and conventions. A valued field will be construed as an $\mathcal{L}_{\preccurlyeq}$-structure in the usual way.

Let K be a valued field. We denote its valuation ring by R (by R_F if we are dealing with a valued field F instead). Let $\smallo(R)$ be the maximal ideal of R and $\boldsymbol{k} := R/\smallo(R)$ the residue field of K. We also let $v : K^\times \to \Gamma$ with $\Gamma = v(K^\times)$ be a valuation on the field K such that $R = \{z \in K : v(z) \geqslant 0\}$ (and if we are dealing instead with a valued field F, we have likewise the residue field $\boldsymbol{k}_F$ and a valuation $v_F : F^\times \to \Gamma_F$).

A *coefficient field of* K is a lift of $\boldsymbol{k}$, that is, a subfield C of K such that $C \subseteq R$ and C maps bijectively onto $\boldsymbol{k}$ under the residue map $R \to \boldsymbol{k}$, equivalently, a subfield C of K such that $R = C + \smallo(R)$. Likewise, a *monomial group* of K is a lift of Γ, that is, a subgroup G of $K^\times$ that is mapped bijectively onto Γ by $v : K^\times \to \Gamma$. If K is henselian (by which we mean that the local ring R is henselian) and $\boldsymbol{k}$ has characteristic 0, then K has a coefficient field; see for example [29, Lemma 2.9]. If K is algebraically closed or $\aleph_1$-saturated, then K has a monomial group; see for example [2, Lemmas 3.3.32, 3.3.39].

Let G be a monomial group of K. Then we can introduce an "absolute value" function $|\cdot|_G : K \to K$ as follows: $|0|_G = 0$, and for $a \in K^\times$,

$$|a|_G = g, \qquad \text{where } g \in G,\ a \in gR^\times.$$

This function is definable in the expansion (K, G) of the valued field K, takes values in $G \cup \{0\}$, and is the identity on $G \cup \{0\}$. Let in addition C be a coefficient field of K. Then we introduce an "angular component" map $\mathrm{ac} : K \to K$ as follows: $\mathrm{ac}(0) := 0$, and for $a \in K^\times$,

$$\mathrm{ac}(a) \;=\; c, \qquad \text{with } c \in C^\times \text{ such that } a \in |a|_G\big(c + \mathcal{o}(R)\big).$$

This function is definable in the expansion (K, C, G) of the valued field K, takes values in C and is the identity on C.

Throughout A is as in Section 10.3: A is noetherian with an ideal $\mathcal{o}(A) \neq A$, such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ and A is $\mathcal{o}(A)$-adically complete.

By an *A-field* we mean a valued field F whose valuation ring R_F is equipped with an A-analytic structure; we construe an A-field as an $\mathcal{L}^A_{\preccurlyeq,D}$-structure in the obvious way. Thus any A-extension of F is an A-field. By an *Acg-field* we mean an expansion $\mathcal{F} = (F, C, G)$ of an A-field F where C is (the underlying set of) a coefficient field of F and G is (the underlying set of) a monomial group G of F. Let $\mathcal{L}^{Acg}_{\preccurlyeq,D}$ be the language $\mathcal{L}^A_{\preccurlyeq,D}$ augmented by unary predicate symbols C and G. We construe an *Acg*-field as an $\mathcal{L}^{Acg}_{\preccurlyeq,D}$-structure in the obvious way.

Example to keep in mind: $\mathcal{F} = \big(F, C, t^{\mathbb{Z}}\big)$, where C is any field, F is the Laurent series field $C((t))$ with valuation ring $C[[t]]$, and $A = C[[t]]$, $\mathcal{o}(A) = tC[[t]]$, with the natural A-analytic structure on $C[[t]]$. To simplify notation we denote this *Acg*-field $\mathcal{F}$ by $\big(C((t)), C, t^{\mathbb{Z}}\big)$.

In the rest of this section $\mathcal{K} = (K, C, G)$ is an Acg-field such that the valuation A-ring R of K is viable; $\boldsymbol{k}$ is the residue field of K and $\Gamma := v(K^\times)$ its value group.

Good substructures and good maps. Our aim is to establish an analogue of the Equivalence Theorem [29, 5.21] in our *analytic* setting *with coefficient field and monomial group*, and we follow the general setup and proof strategy there.

A *good substructure* of $\mathcal{K}$ is an $\mathcal{L}^{Acg}_{\preccurlyeq,D}$-substructure $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ of $\mathcal{K}$ which is also an *Acg*-field. Note that then E is an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of K, and

$$C_{\mathcal{E}} \;=\; C \cap E \;=\; \mathrm{ac}(E), \qquad G_{\mathcal{E}} \;=\; G \cap E \;=\; |E^\times|_G.$$

Below, $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ is a good substructure of $\mathcal{K}$. By Lemma 10.27 the valuation A-ring R_E is viable. For $a \in K$, set $\mathcal{E}_a := \big(E_a, C \cap E_a, G \cap E_a\big) \subseteq \mathcal{K}$.

Lemma 12.13. *We consider four cases for an element $a \in C \cup G$:*

- *(i)* $a \in C$ is algebraic over E. *Then a is algebraic over $C_{\mathcal{E}}$, $E[a]$ is the underlying field of E_a, $C \cap E_a = C_{\mathcal{E}}[a]$, and $G \cap E_a = G_{\mathcal{E}}$;*
- *(ii)* C is infinite and $a \in C$ is transcendental over E. *Then E_a is an immediate extension of $E(a)$, $C \cap E_a = C_{\mathcal{E}}(a)$, and $G \cap E_a = G_{\mathcal{E}}$;*
- *(iii)* $a \in G$ is algebraic over E. *Then $a^d \in G_{\mathcal{E}}$ for some $d \geqslant 1$, $E[a]$ is the underlying field of E_a, $C \cap E_a = C_{\mathcal{E}}$, and $G \cap E_a = G_{\mathcal{E}} \cdot a^{\mathbb{Z}}$;*
- *(iv)* $v(E^\times)$ is a $\mathbb{Z}$-group, $a \in G$, and a is transcendental over E. *Then E_a is an immediate extension of $E(a)$, $C \cap E_a = C_{\mathcal{E}}$, and $G \cap E_a = G_{\mathcal{E}} \cdot a^{\mathbb{Z}}$.*

In each of these four cases, $\mathcal{E}_a$ is a good substructure of $\mathcal{K}$.

Proof. If $a \in C \cup G$ is algebraic over E, this follows from Corollary 12.3. In the transcendental case, use Propositions 12.10 and 12.12. $\square$

In this subsection, $\mathcal{K}' = (K', C', G')$ is an *Acg*-field like $\mathcal{K}$: its valuation A-ring R' is viable. We also let $\mathcal{E}' = (E', C_{\mathcal{E}'}, G_{\mathcal{E}'})$ be a good substructure of $\mathcal{K}'$, and for $a' \in K'$ we set $\mathcal{E}'_{a'} := \big(E'_{a'}, C' \cap E'_{a'}, G' \cap E'_{a'}\big)$.

Let $\mathcal{L}_r := \{0, 1, +, -, \cdot\}$ be the language of rings and $\mathcal{L}_v := \{1, \cdot, \preccurlyeq\}$ the language of (multiplicative) ordered abelian groups, taken as sublanguages of $\mathcal{L}^{A}_{\preccurlyeq, D}$; we construe C, C' as $\mathcal{L}_r$-structures and G, G' as $\mathcal{L}_v$-structures accordingly.

A *good map* $f : \mathcal{E} \to \mathcal{E}'$ is an $\mathcal{L}^{Acg}_{\preccurlyeq, D}$-isomorphism $\mathcal{E} \to \mathcal{E}'$ such that:

(r) the $\mathcal{L}_r$-isomorphism $f|_{C_{\mathcal{E}}} : C_{\mathcal{E}} \to C_{\mathcal{E}'}$ is a partial elementary map from C to C';

(v) the $\mathcal{L}_v$-isomorphism $f|_{G_{\mathcal{E}}} : G_{\mathcal{E}} \to G_{\mathcal{E}'}$ is a partial elementary map from G to G'.

Theorem 12.14. *Suppose* $\operatorname{char} \boldsymbol{k} = 0$ *and Γ is a $\mathbb{Z}$-group. Let $f : \mathcal{E} \to \mathcal{E}'$ be a good map. Then f is a partial elementary map from $\mathcal{K}$ to $\mathcal{K}'$.*

Proof. By passing to suitable elementary extensions we can and do assume that the underlying valued fields of $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal greater than the cardinalities of $C_{\mathcal{E}}$ and $G_{\mathcal{E}}$. A good substructure

$$\mathcal{E}_1 \;=\; (E_1, C_{\mathcal{E}_1}, G_{\mathcal{E}_1})$$

of $\mathcal{K}$ is termed *small* if κ is greater than the cardinalities of $C_{\mathcal{E}_1}$ and $G_{\mathcal{E}_1}$. We shall prove that for any $a \in \mathcal{K}$ we can extend f to a good map with small domain $\mathcal{F} \supseteq \mathcal{E}$ such that $a \in \mathcal{F}$. By the properties of "back-and-forth" this suffices. In addition to Corollary 10.41, we will need the extension procedures in (1)–(4) below.

In (1) and (2) we assume $a \in C$ and extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{F}$ of $\mathcal{K}$ and $\mathcal{F}'$ of $\mathcal{K}'$ and our good map f to a good map $g : \mathcal{F} \to \mathcal{F}'$ such that $a \in C_{\mathcal{F}}$ and $G_{\mathcal{E}} = G_{\mathcal{F}}$. In (3) and (4) we assume (among other things) that $a \in G$, and extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{F}$ of $\mathcal{K}$ and $\mathcal{F}'$ of $\mathcal{K}'$ and our good map f to a good map $g : \mathcal{F} \to \mathcal{F}'$ such that $a \in G_{\mathcal{F}}$ and $C_{\mathcal{E}} = C_{\mathcal{F}}$.

(1) *The case that $a \in C$ is algebraic over E.* By κ-saturation of $\mathcal{K}'$ we obtain an $\mathcal{L}_r$-isomorphism $g_r : C_{\mathcal{E}}[a] \to C_{\mathcal{E}'}[a']$ extending $f|_{C_{\mathcal{E}}}$ and sending a to a' such that g_r is a partial elementary map from C to C'.

Now [29, Lemma 3.21] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $g : E[a] \to E'[a']$ extending both f and g_r. Then g is an $\mathcal{L}^{A}_{\preccurlyeq, D}$-isomorphism $E_a \to E'_{a'}$ by Corollary 12.3 and Proposition 12.9. By Lemma 12.13 (i), $\mathcal{E}_a$ and $\mathcal{E}'_{a'}$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

(2) *The case that $a \in C$ is transcendental over E.* As in (1) we have $a' \in C'$ and an $\mathcal{L}_r$-isomorphism $g_r : C_{\mathcal{E}}(a) \to C_{\mathcal{E}'}(a')$ extending $f|_{C_{\mathcal{E}}}$ and sending a to a' such that g_r is a partial elementary map from C to C'.

Then [29, Lemma 3.22] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $E(a) \to E'(a')$ extending both f and g_r. This $\mathcal{L}_{\preccurlyeq}$-isomorphism extends to an $\mathcal{L}^{A}_{\preccurlyeq, D}$-isomorphism $g :\; E_a \to E'_{a'}$ by Proposition 12.9. Lemma 12.13(ii) gives that $\mathcal{E}_a$ and $\mathcal{E}'_{a'}$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and that g is a good map.

(3) *The case that $a \in G \setminus G_{\mathcal{E}}$ and $a^p \in G_{\mathcal{E}}$, where p is a prime number.* As before we get $a' \in G'$ and an $\mathcal{L}_v$-isomorphism $g_v : G_{\mathcal{E}} \cdot a^{\mathbb{Z}} \to G_{\mathcal{E}'} \cdot a'^{\mathbb{Z}}$ extending $f|_{G_{\mathcal{E}}}$ and sending a to a' such that g_v is a partial elementary map from G to G'. Now [29, Lemma 5.6] gives an $L_{\preccurlyeq}$-isomorphism $g : E(a) \to E'(a')$ extending

both f and g_v. Then g is an $\mathcal{L}^A_{\preccurlyeq,D}$-isomorphism $E_a \to E'_{a'}$ by Corollary 12.3 and Proposition 12.9. By Lemma 12.13(iii), $\mathcal{E}_a$ and $\mathcal{E}'_{a'}$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

(4) *The case that $a \in G$ and $a^d \notin G_{\mathcal{E}}$ for all $d \geqslant 1$. Here we also assume that $v(E^\times)$ is a $\mathbb{Z}$-group.* As before we get $a' \in G'$ and an $\mathcal{L}_v$-isomorphism $g_v : G_{\mathcal{E}} \cdot a^{\mathbb{Z}} \to G_{\mathcal{E}'} \cdot a'^{\mathbb{Z}}$ extending $f|_{G_{\mathcal{E}}}$ and sending a to a' such that g_v is a partial elementary map from G to G'. Note that a is transcendental over E by [29, Proposition 3.19]; likewise, a' is transcendental over E'.

Now [29, Lemma 3.23] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $E(a) \to E'(a')$ extending both f and g_v. This $\mathcal{L}_{\preccurlyeq}$-isomorphism extends to an $\mathcal{L}^A_{\preccurlyeq,D}$-isomorphism $g:\ E_a \to E'_{a'}$ by Proposition 12.9. By Lemma 12.13(iv), $\mathcal{E}_a$ and $\mathcal{E}'_{a'}$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

We did not require for our initial $\mathcal{E}$ that $v(E^\times)$ is a $\mathbb{Z}$-group. But the ambient value group Γ is a $\mathbb{Z}$-group, so for $\Delta := v(E^\times) \subseteq \Gamma$ and setting

$$\widetilde{\Delta} \ := \ \{\gamma \in \Gamma :\ d\gamma \in \Delta \text{ for some } d \geqslant 1\}$$

we obtain an ordered subgroup $\widetilde{\Delta}$ of Γ that is a $\mathbb{Z}$-group with the same least positive element as Γ and with the same cardinality as Δ. Thus we can use (3) iteratively and form a directed union to extend our initial $\mathcal{E}$ and $\mathcal{E}'$ to good substructures $\mathcal{F}$ of $\mathcal{K}$ and $\mathcal{F}'$ of $\mathcal{K}'$ and our initial good map f to a good map $g : \mathcal{F} \to \mathcal{F}'$ such that $v(F^\times) = \widetilde{\Delta}$ (a $\mathbb{Z}$-group) and $C_{\mathcal{F}} = C_{\mathcal{E}}$. Note also that $\widetilde{\Delta}$ is the smallest (under inclusion) ordered subgroup of Γ with the same least element of Γ that contains Δ and is a $\mathbb{Z}$-group. Moreover, the value group of the domain of the good map g obtained in (4) is $\Delta \oplus \mathbb{Z}va$, which is not a $\mathbb{Z}$-group.

Let now any $a \in K$ be given. Let C_1 be the subfield of C such that $\operatorname{res} C_1 = \operatorname{res} E_a$, and let G_1 be the subgroup of G such that $v(G_1) = \widetilde{\Delta}_a$ where $\Delta_a := v(E_a^\times)$. We do not guarantee that $C_1 \subseteq E_a$ or $G_1 \subseteq E_a^\times$, but $C_{\mathcal{E}}$ and C_1 have the same cardinality, and so do $G_{\mathcal{E}}$ and G_1, by Corollary 12.5 and the remarks above. Thus by iterating (1)–(4), we extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{E}_1 = (E_1, C_1, G_1)$ of $\mathcal{K}$ and $\mathcal{E}'_1 = (E'_1, C'_1, G'_1)$ of $\mathcal{K}'$, and extend f to a good map $f_1 : \mathcal{E}_1 \to \mathcal{E}'_1$. Next, let C_2 be the subfield of C such that $\operatorname{res} C_2 = \operatorname{res} E_{1,a}$, and let G_2 be the subgroup of G such that $v(G_2) = \widetilde{\Delta}_{1,a}$ where $\Delta_{1,a} := v(E_{1,a}^\times)$, and obtain likewise $\mathcal{E}_2 = (E_2, C_2, G_2)$ with $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{K}$, and an extension of f_1 to a good map $f_2 : \mathcal{E}_2 \to \mathcal{E}'_2$, with $\mathcal{E}'_1 \subseteq \mathcal{E}'_2 \subseteq \mathcal{K}'$. Continuing this way we obtain for each n small good substructures

$$\mathcal{E}_n \ = \ (E_n, C_n, G_n) \ \subseteq \ \mathcal{E}_{n+1} \ = \ (E_{n+1}, C_{n+1}, G_{n+1})$$

of $\mathcal{K}$ such that $\operatorname{res} C_{n+1} = \operatorname{res} E_{n,a}$ and $v(G_{n+1}) = \widetilde{\Delta}_{n,a}$ where $\Delta_{n,a} = v(E_{n,a}^\times)$, and small good substructures $\mathcal{E}'_n \subseteq \mathcal{E}'_{n+1}$ of $\mathcal{K}'$, and good maps

$$f_n \ :\ \mathcal{E}_n \to \mathcal{E}'_n, \qquad f_{n+1} \ :\ \mathcal{E}_{n+1} \to \mathcal{E}'_{n+1}$$

such that f_{n+1} extends f_n; here $\mathcal{E}_0 := \mathcal{E}$, $\mathcal{E}'_0 := \mathcal{E}'$ and $f_0 := f$. Then

$$\mathcal{E}_\infty \ := \ \bigcup_n \mathcal{E}_n \ = \ (E_\infty, C_\infty, G_\infty)$$

is a small good substructure of $\mathcal{K}$, and $\mathcal{E}'_\infty := \bigcup_n \mathcal{E}'_n = (E'_\infty, C'_\infty, G'_\infty)$ is a small good substructure of $\mathcal{K}'$, and we have a good map $f_\infty : \mathcal{E}_\infty \to \mathcal{E}'_\infty$ extending each f_n. Using $E_{\infty,a} = \bigcup_n E_{n,a}$ we see that $E_{\infty,a}$ is an immediate extension of E_∞.

If $a \in E_\infty$ we have achieved our goal of extending f to a good map with small domain containing a, so assume $a \notin E_\infty$. Replacing a by a^{-1} if necessary we arrange $a \preccurlyeq 1$. Take a divergent pc-sequence (a_ρ) in E_∞ such that all $a_\rho \preccurlyeq 1$ and $a_\rho \rightsquigarrow a$. Then $(a'_\rho) := \big(f_\infty(a_\rho)\big)$ is a divergent pc-sequence in E'_∞. Since the underlying valued field of $\mathcal{K}'$ is κ-saturated and the cardinality of the value group of E'_∞ is less than κ we have $a' \in K'$ such that $a'_\rho \rightsquigarrow a'$. Note that (a_ρ) is of transcendental type over E_∞, by [29, 4.22, 4.16]. Hence (a'_ρ) is of transcendental type over E'_∞, and so $E'_{\infty,a'}$ is an immediate extension of E'_∞ by Proposition 10.40. This yields the (small) good substructures $\mathcal{E}_{\infty,a} := \big(E_{\infty,a}, C_\infty, G_\infty\big)$ of $\mathcal{K}$ and $\mathcal{E}'_{\infty,a'} := \big(E'_{\infty,a'}, C'_\infty, G'_\infty\big)$ of $\mathcal{K}'$. Moreover, f_∞ extends by Corollary 10.41 to a good map $\mathcal{E}_{\infty,a} \to \mathcal{E}'_{\infty,a'}$, and we have achieved our goal. ∎

Corollary 12.15. *Suppose* $\operatorname{char} \boldsymbol{k} = 0$, Γ *is a* $\mathbb{Z}$*-group,* $C_{\mathcal{E}} \preccurlyeq C$ *as* $\mathcal{L}_r$*-structures, and* $G_{\mathcal{E}} \preccurlyeq G$ *as* $\mathcal{L}_v$*-structures. Then* $\mathcal{E} \preccurlyeq \mathcal{K}$.

Proof. Note that $\mathcal{E}$ is a good substructure of both $\mathcal{K}$ and $\mathcal{K}' := \mathcal{E}$, and the identity on $\mathcal{E}$ is a good map. Now apply Theorem 12.14. ∎

Induced structure on coefficient field and monomial group. In this subsection we assume for our *Acg*-field $\mathcal{K} = (K, C, G)$ that $\operatorname{char} \boldsymbol{k} = 0$ and Γ is a $\mathbb{Z}$-group. Our aim here is Corollary 12.17 on the structure that $\mathcal{K}$ induces on C and G combined. It will be derived in a familiar way from Theorem 12.14 and a fact implicit in its proof. To state that fact we let $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ and $\mathcal{F} = (F, C_{\mathcal{F}}, G_{\mathcal{F}})$ be *Acg*-fields and $\mathcal{L}^{Acg}_{\preccurlyeq,D}$-extensions of $\mathcal{K}$. For $a \in E^n$, let $\operatorname{tp}(a|K)$ be the $\mathcal{L}^{Acg}_{\preccurlyeq,D}$-type of a over K, that is, the set of $\mathcal{L}^{Acg,K}_{\preccurlyeq,D}$-formulas $\phi(Y_1,\dots,Y_n)$ such that $\mathcal{E} \models \phi(a)$. Likewise, for $c \in C^n_{\mathcal{E}}$, let $\operatorname{tp}(c|C)$ be the $\mathcal{L}_r$-type of c over C, and for $g \in G^n_{\mathcal{E}}$, let $\operatorname{tp}(g|G)$ be the $\mathcal{L}_v$-type of g over G.

Lemma 12.16. *Suppose* $\mathcal{E}$ *and* $\mathcal{F}$ *are elementary extensions of* $\mathcal{K}$. *Let* $c_{\mathcal{E}} \in C^m_{\mathcal{E}}$, $g_{\mathcal{E}} \in G^n_{\mathcal{F}}$ *and* $c_{\mathcal{F}} \in C^m_F$, $g_{\mathcal{F}} \in G^n_{\mathcal{F}}$ *be such that*

$$\operatorname{tp}(c_{\mathcal{E}}|C) \;=\; \operatorname{tp}(c_{\mathcal{F}}|C), \qquad \operatorname{tp}(g_{\mathcal{E}}|G) \;=\; \operatorname{tp}^G(g_{\mathcal{F}}|G).$$

Then for the points $(c_{\mathcal{E}}, g_{\mathcal{E}}) \in E^{m+n}$ *and* $(c_{\mathcal{F}}, g_{\mathcal{F}}) \in F^{m+n}$ *we have*

$$\operatorname{tp}\big((c_{\mathcal{E}}, g_{\mathcal{E}})|K\big) \;=\; \operatorname{tp}\big((c_{\mathcal{F}}, g_{\mathcal{F}})|K\big).$$

Proof. By our assumptions $\mathcal{K}$ is a good substructure of both $\mathcal{E}$ and $\mathcal{F}$, and the identity on $\mathcal{K}$ is a good map. Using $\operatorname{tp}(c_{\mathcal{E}}|C) = \operatorname{tp}(c_{\mathcal{F}}|C)$ and the extension procedures (1) and (2) in the proof of Theorem 12.14 in conjunction with Lemma 12.13(i), (ii) we obtain a good map whose domain contains the elements of K and the components of $c_{\mathcal{E}}$ and that is the identity on $\mathcal{K}$ and sends $c_{\mathcal{E}}$ to $c_{\mathcal{F}}$, such that the monomial group of its domain is still $G_{\mathcal{E}}$. Next we use likewise the extension procedures from (3) and (4) in that proof to extend this good map further so that its domain now contains the components of $g_{\mathcal{E}}$ as well, and sends sends $g_{\mathcal{E}}$ to $g_{\mathcal{F}}$. It remains to use Theorem 12.14. ∎

Corollary 12.17. *Each subset of* $C^m \times G^n \subseteq K^{m+n}$ *which is definable in* $\mathcal{K}$ *is a finite union of "rectangles"* $P \times Q$ *with* $P \subseteq C^m$ *definable in the* $\mathcal{L}_r$*-structure* C *and* $Q \subseteq G^n$ *definable in the* $\mathcal{L}_v$*-structure* G.

Proof. Apply Lemma 12.16 in conjunction with [29, Lemmas 5.13, 5.14]. ∎

Corollary 12.18. *If* $P \subseteq K^n$ *is definable in* $\mathcal{K}$, *then* $P \cap C^n$ *is definable in the* $\mathcal{L}_r$*-structure* C, *and* $P \cap G^n$ *is definable in the* $\mathcal{L}_v$*-structure* G.

How does the above relate to the Binyamini-Cluckers-Novikov result? We construe $\mathbb{C}((t))$ below as an A-field in the usual way, with $A = \mathbb{C}[[t]], \smallo(A) = tA$. Proposition 2 in [12] concerns the 3-sorted structure $\mathcal{M}$ consisting of the following:

the A-field $\mathbb{C}((t))$, the field $\mathbb{C}$, the ordered abelian group $\mathbb{Z}$,

(each a 1-sorted structure) and two functions relating the three sorts: the obvious t-adic valuation $v : \mathbb{C}((t))^{\times} \to \mathbb{Z}$, and the "reduced angular component map" $\overline{ac} : \mathbb{C}((t)) \to \mathbb{C}$ that assigns to each nonzero Laurent series $f = \sum_{k\in\mathbb{Z}} c_k t^k$ (all $c_k \in \mathbb{C}$) its leading coefficient $c_{v(f)}$, with $\overline{ac}(0) := 0$ by convention.

This 3-sorted $\mathcal{M}$ should not be confused with the 1-sorted $\big(\mathbb{C}((t)), \mathbb{C}, t^{\mathbb{Z}}\big)$ that is among the Acg-fields $\mathcal{K}$ considered in this section. The natural interpretation of $\mathcal{M}$ in $\big(\mathbb{C}((t)), \mathbb{C}, t^{\mathbb{Z}}\big)$ shows that if a set $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then it is definable in $\big(\mathbb{C}((t)), \mathbb{C}, t^{\mathbb{Z}}\big)$. The converse fails: the sets $\mathbb{C}, t^{\mathbb{Z}} \subseteq \mathbb{C}((t))$ are definable in the latter but not in the former (see [27, Theorem 3.9]); thus the latter is "richer" than the former.

For $d \geqslant 1$ we let $\mathbb{C}[t]_{<d}$ be the set of polynomials in $\mathbb{C}[t]$ of degree $< d$. Then $\mathbb{C}[t]_{<d}$ is a subset of $\mathbb{C}[[t]]$, and thus of $\mathbb{C}((t))$. We identify $\mathbb{C}[t]_{<d}$ with $\mathbb{C}^d$ via the bijection $c_0 + c_1 t + \cdots + c_{d-1}t^{d-1} \mapsto (c_0, \ldots, c_{d-1})$ for $c_0, \ldots, c_{d-1} \in \mathbb{C}$. For $P \subseteq \mathbb{C}((t))^n$ we set $P(d) := P \cap (\mathbb{C}[t]_{<d})^n$, which under the identification above becomes a subset of $\mathbb{C}^{dn}$. Now Proposition 2 in [12] says:

if $P \subseteq \mathbb{C}((t))^n$ is definable in $\mathcal{M}$, then for each $d \geqslant 1$ the set $P(d) \subseteq \mathbb{C}^{dn}$ is a constructible subset of the space $\mathbb{C}^{dn}$ with its Zariski topology.

By "Chevalley-Tarski" a subset of $\mathbb{C}^m$ is constructible iff it is definable in the field $\mathbb{C}$, so this proposition is for $\mathcal{K} = \big(\mathbb{C}((t)), \mathbb{C}, t^{\mathbb{Z}}\big)$ a special case of Corollary 12.18.

References

[1] H. Adler, *Strong theories, burden, and weight*, Preprint (2007), logic.univie.ac.at/~adler/docs/strong.pdf.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017. MR 3585498

[3] J. Ax and S. Kochen, *Diophantine problems over local fields. I*, Amer. J. Math. **87** (1965), 605–630. MR 184930

[4] ———, *Diophantine problems over local fields. II. A complete set of axioms for p-adic number theory*, Amer. J. Math. **87** (1965), 631–648. MR 184931

[5] ———, *Diophantine problems over local fields. III. Decidable fields*, Ann. of Math. (2) **83** (1966), 437–456. MR 201378

[6] B. Bakker, Y. Brunebarbe, and J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, ArXiv e-prints (2018), arxiv.org/abs/1811.12230.

[7] P. Bateman, C. Jockusch, and A. Woods, *Decidability and undecidability of theories with a predicate for the primes*, J. Symbolic Logic **58** (1993), no. 2, 672–687.

[8] O. Belegradek, Y. Peterzil, and F. Wagner, *Quasi-o-minimal structures*, J. Symbolic Logic **65** (2000), no. 3, 1115–1132. MR 1791366

[9] N. Bhardwaj and L. van den Dries, *On the Pila-Wilkie theorem*, Expositiones Mathematicae (2022), https://doi.org/10.1016/j.exmath.2022.03.001.

[10] N. Bhardwaj and C.-M. Tran, *The additive groups of $\mathbb{Z}$ and $\mathbb{Q}$ with predicates for being square-free*, J. Symb. Log. **86** (2021), no. 4, 1324–1349. MR 4362916

[11] E. Bierstone and P. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 5–42. MR 972342

[12] G. Binyamini, R. Cluckers, and D. Novikov, *Point counting and Wilkie's conjecture for non-archimedean Pfaffian and Noetherian functions*, Duke Math. J. (to appear) (2020), arxiv.org/abs/2009.05480.

[13] G. Binyamini and D. Novikov, *Wilkie's conjecture for restricted elementary functions*, Ann. of Math. (2) **186** (2017), no. 1, 237–275. MR 3665004

[14] ______, *Complex cellular structures*, Ann. of Math. (2) **190** (2019), no. 1, 145–248. MR 3990603

[15] ______, *The Yomdin-Gromov algebraic lemma revisited*, Arnold Math. J. **7** (2021), no. 3, 419–430. MR 4293872

[16] G. Binyamini, D. Novikov, and B. Zack, *Wilkie's conjecture for pfaffian structures*, ArXiv e-prints (2022), arxiv.org/abs/2202.05305.

[17] E. Bombieri and J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J. **59** (1989), no. 2, 337–357. MR 1016893

[18] A. Chernikov, *Theories without the tree property of the second kind*, Ann. Pure Appl. Logic **165** (2014), no. 2, 695–723.

[19] A. Chernikov, D. Palacin, and K. Takeuchi, *On n-dependence*, Notre Dame J. Form. Log. **60** (2019), no. 2, 195–214. MR 3952231

[20] R. Cluckers and L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 4, 1147–1223. MR 2800487

[21] ______, *Strictly convergent analytic structures*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 1, 107–149. MR 3584560

[22] G. Conant, *There are no intermediate structures between the group of integers and Presburger arithmetic*, J. Symb. Log. **83** (2018), no. 1, 187–207. MR 3796282

[23] J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Ann. of Math. (2) **128** (1988), no. 1, 79–138. MR 951508

[24] A. Dolich and J. Goodrick, *Strong theories of ordered Abelian groups*, Fund. Math. **236** (2017), no. 3, 269–296.

[25] L. van den Dries, *Remarks on Tarski's problem concerning* $(\mathbb{R}, +, \cdot, \exp)$, Logic colloquium '82 (Florence, 1982), Stud. Logic Found. Math., vol. 112, North-Holland, Amsterdam, 1984, pp. 97–121. MR 762106

[26] ______, *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results*, Bull. Amer. Math. Soc. (N.S.) **15** (1986), no. 2, 189–193. MR 854552

[27] ______, *Analytic Ax-Kochen-Ersov theorems*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), Contemp. Math., vol. 131, Amer. Math. Soc., Providence, RI, 1992, pp. 379–398. MR 1175894

[28] ______, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998. MR 1633348

[29] ______, *Lectures on the model theory of valued fields*, Model theory in algebra, analysis and arithmetic, Lecture Notes in Math., vol. 2111, Springer, Heidelberg, 2014, pp. 55–157. MR 3330198

[30] L. van den Dries, D. Haskell, and D. Macpherson, *One-dimensional p-adic subanalytic sets*, J. London Math. Soc. (2) **59** (1999), no. 1, 1–20. MR 1688485

[31] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. (2) **140** (1994), no. 1, 183–205. MR 1289495

[32] L. van den Dries and C. Miller, *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), no. 1-3, 19–56. MR 1264338

[33] Ju. L. Eršov, *On elementary theories of local fields*, Algebra i Logika Sem. **4** (1965), no. 2, 5–30. MR 0207686

[34] ———, *On elementary theory of maximal normalized fields*, Algebra i Logika Sem. **4** (1965), no. 3, 31–70. MR 0193086

[35] ———, *On the elementary theory of maximal normed fields. II*, Algebra i Logika Sem. **5** (1966), no. 1, 5–40. MR 0204290

[36] ———, *On the elementary theory of maximal normed fields. III*, Algebra i Logika Sem. **6** (1967), no. 3, 31–38. MR 0223231

[37] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston, Inc., Boston, MA, 2004. MR 2014891

[38] H. Gonshor, *An introduction to the theory of surreal numbers*, London Mathematical Society Lecture Note Series, vol. 110, Cambridge University Press, Cambridge, 1986. MR 872856

[39] M. Gromov, *Entropy, homology and semialgebraic geometry*, Astérisque (1987), no. 145-146, 5, 225–240, Séminaire Bourbaki, Vol. 1985/86. MR 880035

[40] Y. Gurevich and P. H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 171–182. MR 742419

[41] I. Kaplan and S. Shelah, *Decidability and classification of the theory of integers with primes*, J. Symb. Log. **82** (2017), no. 3, 1041–1050. MR 3694340

[42] B. Kim, *Simplicity theory*, Oxford Logic Guides, vol. 53, Oxford University Press, Oxford, 2014. MR 3156332

[43] B. Kim, H.-J. Kim, and L. Scow, *Tree indiscernibilities, revisited*, Arch. Math. Logic **53** (2014), no. 1-2, 211–232. MR 3151406

[44] L. Lipshitz, *Rigid subanalytic sets*, Amer. J. Math. **115** (1993), no. 1, 77–108. MR 1209235

[45] L. Lipshitz and Z. Robinson, *Rigid subanalytic subsets of curves and surfaces*, J. London Math. Soc. (2) **59** (1999), no. 3, 895–921. MR 1709087

[46] Ju. V. Matijasevič, *The Diophantineness of enumerable sets*, Dokl. Akad. Nauk SSSR **191** (1970), 279–282. MR 0258744

[47] C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 257–259. MR 1195484

[48] L. Mirsky, *Note on an asymptotic formula connected with r-free integers*, Quart. J. Math., Oxford Ser. **18** (1947), 178–182.

[49] Y. Peterzil and S. Starchenko, *Complex analytic geometry in a nonstandard setting*, Model theory with applications to algebra and analysis. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 349, Cambridge Univ. Press, Cambridge, 2008, pp. 117–165. MR 2441378

[50] J. Pila, *Integer points on the dilation of a subanalytic surface*, Q. J. Math. **55** (2004), no. 2, 207–223. MR 2068319

[51] ———, *On the algebraic points of a definable set*, Selecta Math. (N.S.) **15** (2009), no. 1, 151–170. MR 2511202

[52] ———, *O-minimality and the André-Oort conjecture for $\mathbb{C}^n$*, Ann. of Math. (2) **173** (2011), no. 3, 1779–1840. MR 2800724

[53] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616. MR 2228464

[54] J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **19** (2008), no. 2, 149–162. MR 2411018

[55] A. Pillay and C. Steinhorn, *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 565–592. MR 833697

[56] A. Robinson, *On ordered fields and definite functions*, Math. Ann. **130** (1955), 275–271. MR 75932

[57] K. Rogers, *The Schnirelmann density of the squarefree integers*, Proc. Amer. Math. Soc. **15** (1964), 515–516.

[58] J.-P. Rolin, P. Speissegger, and A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), no. 4, 751–777. MR 1992825

[59] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 54, 323–401. MR 644559

[60] A. Tarski, *A decision method for elementary algebra and geometry*, University of California Press, Berkeley-Los Angeles, Calif., 1951, 2nd ed. MR 0044472

[61] C.-M. Tran, *Tame structures via multiplicative character sums on varieties over finite fields*, ArXiv e-prints (2017), arxiv.org/abs/1704.03853.

[62] F. Wagner, *Stable groups*, London Mathematical Society Lecture Note Series, vol. 240, Cambridge University Press, Cambridge, 1997. MR 1473226

[63] A. J. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no. 4, 1051–1094. MR 1398816

[64] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), no. 3, 285–300. MR 889979