ON THE PILA-WILKIE THEOREM

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Abstract. This expository paper gives an account of the Pila-Wilkie counting theorem and some of its extensions and generalizations. We use semialgebraic cell decomposition to simplify part of the original proof. Included are complete treatments of a result due to Pila and Bombieri and of the o-minimal Yomdin-Gromov theorem that are used in this proof.

1. Introduction and some notations

In these notes we prove the Pila-Wilkie theorem following the original paper [5], but exploiting cell decomposition more thoroughly to simplify the deduction from its main ingredients. By including proofs of these ingredients and adding an Appendix on o-minimality we make this account self-contained, within reason.

We also obtain two generalizations due to Pila [4], one where instead of rational points we count points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ with a finite bound on its dimension, and one where instead we count points with coordinates that are algebraic of at most a given degree over $\mathbb{Q}$. The general approach is as in [4], but the technical details seem to us a bit simpler.

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Throughout, $d, e, k, l, m, n \in \mathbb{N} = \{0, 1, 2, \dots\}$, and $\varepsilon, c, K \in \mathbb{R}^{>} := \{t \in \mathbb{R} : t > 0\}$. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$, and given a field $\boldsymbol{k}$ (often $\boldsymbol{k} = \mathbb{R}$) we set $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ for the usual coordinate functions $x_1, \dots, x_m$ on $\boldsymbol{k}^m$, and likewise $a^\alpha := a_1^{\alpha_1} \cdots a_m^{\alpha_m}$ for any point $a = (a_1, \dots, a_m) \in \boldsymbol{k}^m$. Let $U \subseteq \mathbb{R}^m$ be open. For a function $f : U \to \mathbb{R}$ of class C^k and $\alpha \in \mathbb{N}^m$, $|\alpha| \leqslant k$,

$$f^{(\alpha)} := \frac{\partial^{|\alpha|}}{\partial x^\alpha} f$$

denotes the corresponding partial derivative of order α. We extend the above to C^k-maps $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ by

$$f^{(\alpha)} := (f_1^{(\alpha)}, \dots, f_n^{(\alpha)}) : U \to \mathbb{R}^n$$

for α as before. This includes the case $m = 0$, where $\mathbb{R}^0$ has just one point and any map $f : U \to \mathbb{R}^n$ is of class C^k for all k, with $f^{(\alpha)} = f$ for the unique $\alpha \in \mathbb{N}^0$. For $a_1, \dots, a_n \in \mathbb{R}^{\geqslant} := \{t \in \mathbb{R} : t \geqslant 0\}$ the number $\max\{a_1, \dots, a_n\} \in \mathbb{R}^{\geqslant}$ equals 0 by convention if $n = 0$. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we set $|a| := \max\{|a_1|, \dots, |a_n|\}$ in $\mathbb{R}^{\geqslant}$; this conflicts with our notation $|\alpha|$ for $\alpha \in \mathbb{N}^n$, but in practice no confusion will arise. We also use these notational conventions when instead of $\mathbb{R}$ we have any o-minimal field with U and f definable in it. We include an appendix on the basic facts concerning o-minimality and definability for those not familiar with these topics. Here we just mention that an o-minimal field is by definition an ordered field

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equipped with an o-minimal structure on it; this ordered field is then necessarily real closed: adjoining i with $i^2 = -1$ makes it algebraically closed. An *o-minimal expansion of* $\mathbb{R}$ is an o-minimal field whose underlying ordered field is $\mathbb{R}$.

The Pila-Wilkie theorem and two ingredients of the proof. First some notation needed to state the theorem. We define the *multiplicative height function* $\mathrm{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathrm{H}(\frac{a}{b}) := \max(|a|,\ |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Thus $\mathrm{H}(0) = \mathrm{H}(1) = \mathrm{H}(-1) = 1$, and for $q \in \mathbb{Q}$ we have $\mathrm{H}(q) \geqslant 2$ if $q \notin \{0, 1, -1\}$, $\mathrm{H}(q) = \mathrm{H}(-q)$, and $\mathrm{H}(q^{-1}) = \mathrm{H}(q)$ for $q \neq 0$. For $a = (a_1, ..., a_n) \in \mathbb{Q}^n$,

$$\mathrm{H}(a) := \max\{\mathrm{H}(a_1), \ldots, \mathrm{H}(a_n)\} \in \mathbb{N}.$$

Let $X \subseteq \mathbb{R}^n$. We set $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$. Throughout T ranges over real numbers $\geqslant 1$, and we set $X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) :\ \mathrm{H}(a) \leqslant T\}$ be the (finite) set of rational points of X of height $\leqslant T$, and set $\mathrm{N}(X, T) := \#X(\mathbb{Q}, T) \in \mathbb{N}$. The *algebraic part* of X, denoted by X^{alg}, is the union of the connected infinite semialgebraic subsets of X. So for $n \geqslant 1$, the interior of X (in $\mathbb{R}^n$) is part of X^{alg}.

Example. The set $X := \{(a, b, c) \in \mathbb{R}^3 :\ 1 < a, b < 2,\ c = a^b\}$ is definable in the o-minimal field $\mathbb{R}_{\exp}$. (See the subsection "O-Minimal Structures" in part A of the Appendix for $\mathbb{R}_{\exp}$.) For rational $q \in (1, 2)$, we have a semialgebraic curve

$$X_q := \{(a, q, c) :\ c = a^q\} \subseteq X,$$

and X^{alg} is the union of those X_q (proved at the end of part A of the appendix).

We also set

$$X^{\mathrm{tr}} := X \setminus X^{\mathrm{alg}} \qquad (\textit{the transcendental part of } X).$$

We can now state the Pila-Wilkie theorem, also called the *Counting Theorem*:

Theorem 1.1. *Let $X \subseteq \mathbb{R}^n$ with $n \geqslant 1$ be definable in some o-minimal expansion of $\mathbb{R}$. Then for all ε there is a c such that for all T,*

$$\mathrm{N}(X^{\mathrm{tr}}, T) \leqslant cT^{\varepsilon}.$$

Roughly speaking, it says there are few rational points on the transcendental part of a set definable in an o-minimal expansion of $\mathbb{R}$: the number of such points grows slower than any power T^{ε} with T bounding their height. To apply the counting theorem one needs to describe X^{alg} in some useful way. This typically involves *Ax-Schanuel* type transcendence results.

Note that $X^{\mathrm{tr}}(\mathbb{Q}) = \emptyset$ in the example above, so the theorem is trivial for this X. We shall include a refinement, Theorem 2.5, which is nontrivial for this X.

The proof of Theorem 1.1 depends on two intermediate results. The first of these has nothing to do with o-minimality. To state it we define for $k, n \geqslant 1$ and $X \subseteq \mathbb{R}^n$ a *strong k-parametrization of* X to be a C^k-map $f : (0, 1)^m \to \mathbb{R}^n$, $m < n$, with image X, such that $|f^{(\alpha)}(a)| \leqslant 1$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and all $a \in (0, 1)^m$. We also define a *hypersurface in* $\mathbb{R}^n$ *of degree* $\leqslant e$ to be the zeroset in $\mathbb{R}^n$ of a nonzero polynomial in $x = (x_1, \ldots, x_n)$ over $\mathbb{R}$ of (total) degree $\leqslant e$. The first of these intermediate results is essentially due to Pila and Bombieri, cf. [1, 3].

Theorem 1.2. *Let $n \geqslant 1$ be given. Then for any $e \geqslant 1$ there are $k = k(n,\ e) \geqslant 1$, $\varepsilon = \varepsilon(n,\ e)$, and $c = c(n,\ e)$, such that if $X \subseteq \mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^{ε} many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q}, T)$, with $\varepsilon(n,\ e) \to 0$ as $e \to \infty$.*

We prove this in Section 3. In Section 2 we obtain Theorem 1.1 from Theorem 1.2 by induction on n, using a strong parametrization result. Yomdin [7] and Gromov [2] proved such a strong parametrization uniformly for the members of any semialgebraic family of subsets of $[-1,1]^n$. We need this for any definable "o-minimal" family. To make this precise, let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$. For $s \in E$,

$$X(s) \ := \ \{a \in \mathbb{R}^n : \ (s,a) \in X\} \quad \text{(a } \textit{section} \text{ of } X)$$

We consider E, X as describing the family $\big(X(s)\big)_{s\in E}$ of sections $X(s) \subseteq \mathbb{R}^n$; the sets $X(s)$ are the members of the family. If E and X are definable in the o-minimal expansion $\widetilde{\mathbb{R}}$ of $\mathbb{R}$, then its members are definable in $\widetilde{\mathbb{R}}$.

Theorem 1.3. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of $\mathbb{R}$ and $E \subseteq \mathbb{R}^m$ and $X \subseteq E\times\mathbb{R}^n$ with $n \geqslant 1$ definable in $\widetilde{\mathbb{R}}$ such that every section $X(s)$ is a subset of $[-1,1]^n$ with empty interior. Then there is for every $k \geqslant 1$ an $M \in \mathbb{N}$ such that every section $X(s)$ is the union of at most M subsets, each having a strong k-parametrization.*

This is enough for use in the next section, but the proof in Sections 4–7 gives more precise results. In the appendix we also expose some elementary model theory used in that proof. In Section 8 we treat extensions of the Counting Theorem from [4].

2. Proof of the Counting Theorem from the two ingredients

Throughout this section we assume $n \geqslant 1$. We begin by stating some elementary facts about X^{alg} and X^{tr} for $X \subseteq \mathbb{R}^n$. The first is obvious:

Lemma 2.1. *If $X = X_1 \cup \cdots \cup X_m$, then $X^{\mathrm{alg}} \supseteq X_1^{\mathrm{alg}} \cup \cdots \cup X_m^{\mathrm{alg}}$, and thus*

$$X^{\mathrm{tr}} \ \subseteq \ X_1^{\mathrm{tr}} \cup \cdots \cup X_m^{\mathrm{tr}}.$$

Note also that if X is open in $\mathbb{R}^n$, then $X^{\mathrm{tr}} = \emptyset$.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and f maps the set $X \subseteq S$ homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Then $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$ and thus $f(X^{\mathrm{tr}}) = Y^{\mathrm{tr}}$. (We allow $m = 0$ for later inductions.)*

Proof. It is clear that $f(X^{\mathrm{alg}}) \subseteq Y^{\mathrm{alg}}$. Also, for any connected infinite semialgebraic set $C \subseteq Y$, the set $f^{-1}(C) \subseteq S$ is semialgebraic (since C and f are), contained in X (since f is injective), hence connected and infinite, and thus $f^{-1}(C) \subseteq X^{\mathrm{alg}}$. This shows $f^{-1}(Y^{\mathrm{alg}}) \subseteq X^{\mathrm{alg}}$, and thus $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$. □

In order to apply Theorem 1.3 we need to reduce to the case of subsets of $[-1,1]^n$. This is done as follows. For $X \subseteq \mathbb{R}^n$ and $I \subseteq \{1,\dots,n\}$, set

$$X_I \ := \ \{a \in X : \ |a_i| > 1 \text{ for all } i \in I, \ |a_i| \leqslant 1 \text{ for all } i \notin I\}$$

and define the semialgebraic map $f_I : \mathbb{R}^n_I \to \mathbb{R}^n$ by $f_I(a) = b$ where $b_i := a_i^{-1}$ for $i \in I$ and $b_i := a_i$ for $i \notin I$. Thus f_I maps $\mathbb{R}^n_I$ homeomorphically onto its image, a subset of $[-1,1]^n$. If $I = \emptyset$, then f_I is the inclusion map $\mathbb{R}^n_I = [-1,1]^n \to \mathbb{R}^n$. Note that for $a \in \mathbb{Q}^n$ we have $f_I(a) \in \mathbb{Q}^n$ and $\mathrm{H}(a) = \mathrm{H}\big(f_I(a)\big)$. Moreover, X is the disjoint union of the sets X_I, and for $Y_I = f_I(X_I)$ we have $Y_I \subseteq [-1,1]^n$, $Y_I^{\mathrm{tr}} = f_I(X_I^{\mathrm{tr}})$ by Lemma 2.2, so $\mathrm{N}(Y_I^{\mathrm{tr}}, T) = \mathrm{N}(X_I^{\mathrm{tr}}, T)$ for all T.

The sketch below actually proves the Counting Theorem, modulo a uniformity assumption that arises at the end of the sketch. This motivates a stronger "definable family" version of the theorem, which we then prove as in the sketch.

In the rest of this section we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of $\mathbb{R}$, and *definable* is with respect to $\widetilde{\mathbb{R}}$. We exploit facts about semialgebraic cells $C \subseteq \mathbb{R}^n$ and the corresponding homeomorphisms $p_C : C \to p(C)$; see the subsections "Cells" and "Cell Decomposition" in part A of the Appendix.

Sketch of the proof of Theorem 1.1 from Theorems 1.2 and 1.3. Let $X \subseteq \mathbb{R}^n$ be definable. We need to show that there are 'few' rational points on X outside X^{alg}. We proceed by induction on n. By Lemma 2.1 and the remark following it we can remove the interior of X in $\mathbb{R}^n$ from X and arrange that X has empty interior. As indicated just before this sketch we also arrange $X \subseteq [-1,1]^n$.

Let ε be given, and take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 1.2, and take $k = k(n,e)$. Theorem 1.3 gives $M \in \mathbb{N}$ such that X is a union of at most M subsets, each admitting a strong k-parametrization. Then Theorem 1.2 gives $X(\mathbb{Q},T) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=1}^{J} H_{ij}$, where the H_{ij} are hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$, and $J \in \mathbb{N}$, $J \leqslant cT^{\varepsilon/2}$, $c = c(n,e)$ as in that theorem. If $a \in X^{\text{tr}}(\mathbb{Q},T)$ and $a \in H_{ij}$, then clearly $a \in (X \cap H_{ij})^{\text{tr}}$. Thus

$$X^{\text{tr}}(\mathbb{Q},T) \ \subseteq\ \bigcup_{i=1}^{M} \bigcup_{j=1}^{J} (X \cap H_{ij})^{\text{tr}}(\mathbb{Q},T).$$

Let H be any hypersurface in $\mathbb{R}^n$ of degree $\leqslant e$. We aim for an upper bound on $\mathrm{N}\big((X \cap H)^{\text{tr}}, T\big)$ of the form $c_1 T^{\varepsilon/2}$ with $c_1 \in \mathbb{R}^{>}$ independent of H and T. (If we achieve this, then applying this to the hypersurfaces H_{ij} we obtain

$$\mathrm{N}(X^{\text{tr}},T) \ \leqslant\ MJc_1T^{\varepsilon/2} \ \leqslant\ M \cdot cT^{\varepsilon/2} \cdot c_1 T^{\varepsilon/2} \ =\ Mcc_1 \cdot T^{\varepsilon},$$

and we are done.) Take semialgebraic cells $C_1, \dots, C_L$ in $\mathbb{R}^n$, $L \in \mathbb{N}$, such that

$$H \ =\ C_1 \cup \cdots \cup C_L.$$

Suppose $C = C_l$ is one of those cells. Then we have a semialgebraic homeomorphism $p = p_C : C \to p(C) = p(C_l)$ onto an open cell $p(C_l)$ in $\mathbb{R}^{n_l}$ with $n_l < n$, and so p maps $X \cap C_l$ homeomorphically onto its image $Y_l \subseteq p(C_l) \subseteq \mathbb{R}^{n_l}$. Now p is given by omitting $n - n_l$ of the coordinates, so for $a \in C_l(\mathbb{Q})$ we have $p(a) \in \mathbb{Q}^{n_l}$ and $\mathrm{H}\big(p(a)\big) \leqslant \mathrm{H}(a)$. The hypersurfaces of degree $\leqslant e$ in $\mathbb{R}^n$ belong to a single semialgebraic family, so by Proposition A.4 we can (and do) take here $L \leqslant L(e,n)$, with $L(e,n) \in \mathbb{N}^{\geqslant 1}$ depending only on e, n. By Lemma 2.1,

$$(X \cap H)^{\text{tr}} \ \subseteq\ (X \cap C_1)^{\text{tr}} \cup \cdots \cup (X \cap C_L)^{\text{tr}}.$$

Since the $n_l < n$ we can (and do) assume inductively that for all T,

$$\mathrm{N}(Y_l^{\text{tr}},T) \ \leqslant\ B_l T^{\varepsilon/2}, \quad l = 1, \dots, L$$

with $B_l \in \mathbb{R}^{>}$ independent of T. Hence for all T,

$$\mathrm{N}\big((X \cap C_l)^{\text{tr}}), T\big) \ \leqslant\ B_l T^{\varepsilon/2}, \quad l = 1, \dots, L$$

by Lemma 2.2 applied to the maps $p = p_{C_l}$, and thus

$$\mathrm{N}\big((X \cap H)^{\text{tr}}, T\big) \ \leqslant\ (B_1 + \cdots + B_L)T^{\varepsilon/2}.$$

Assume we can take $B_1, \dots, B_L \leqslant B$ with $B \in \mathbb{R}^{>}$ depending only on X, ε, not on $H, Y_1, \dots, Y_L$. Then $c_1 := L(e,n)B$ is a positive real number as aimed for.

The above sketch is a proof, modulo the assumption at the end. The hypersurfaces H in the sketch belong fortunately to a single semialgebraic family, a fact we already used, and so the sets Y_l as H varies can be taken to belong to a single definable family. The inductive hypothesis should accordingly include this uniformity, and so the full proof should be carried out not just for one set X, but uniformly for all sets from a definable family, with constants depending only on the family. This is why we need Theorem 1.3 not just for a single definable $X \subseteq [-1,1]^n$ but for all members of a definable family of such sets. (As to the M introduced at the beginning of the sketch, Theorem 1.3 also provides an M that works for all members of the family.) Below we carry out the details.

The next lemma is a routine consequence of Theorem A.2 and Proposition A.4. The $\boldsymbol{i}$-cells in this lemma and the projection maps $p_{\boldsymbol{i}} : \mathbb{R}^n \to \mathbb{R}^d$ in the proof of Theorem 2.4 are defined in the subsection "Cells" from part A of the Appendix.

Lemma 2.3. *Let $e \geqslant 1$ and set $D := \binom{e+n}{n}$, the dimension of the $\mathbb{R}$-linear space of polynomials over $\mathbb{R}$ in n variables and of degree $\leqslant e$. Then there are $L \in \mathbb{N}^{\geqslant 1}$ and semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$, $F := \mathbb{R}^D \setminus \{0\}$, such that*

$$\{\mathcal{H}(t) : \ t \in F\} = \text{ set of hypersurfaces in } \mathbb{R}^n \text{ of degree} \leqslant e,$$

$\mathcal{H}(t) = \mathcal{C}_1(t) \cup \dots \cup \mathcal{C}_L(t)$ for all $t \in F$, and for each $l \in \{1, \dots, L\}$ there is an $\boldsymbol{i} = (i_1, \dots, i_n)$ in $\{0,1\}^n$, $\boldsymbol{i} \neq (1, \dots, 1)$, with the property that every $\mathcal{C}_l(t)$ with $t \in F$ is a semialgebraic $\boldsymbol{i}$-cell in $\mathbb{R}^n$ or empty.

Two family versions of the counting theorem. In this subsection we assume that $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

Theorem 2.4. *Let any ε be given. Then there is a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T we have $\mathrm{N}(X(s)^{\mathrm{tr}}, T) \leqslant cT^{\varepsilon}$.*

Proof. We proceed by induction on n. As in the sketch we reduce to the case where $X(s)$ is for every $s \in E$ a subset of $[-1,1]^n$ with empty interior. Take $e \geqslant 1$ so large that $\varepsilon(n,e) \leqslant \varepsilon/2$ in Theorem 1.2, and set $k = k(n,e)$. So for every $Z \subseteq \mathbb{R}^n$ with a strong k-parameterization we can cover $Z(\mathbb{Q}, T)$ with at most $cT^{\varepsilon/2}$ hypersurfaces of degree $\leqslant e$ where $c = c(n,e)$ is as in Theorem 1.2. Theorem 1.3 gives $M \in \mathbb{N}$ such that each section $X(s)$ is a union of at most M subsets, each admitting a strong k-parametrization. Let $s \in E$, and let H be a hypersurface of degree $\leqslant e$. As in the sketch we see that by our choice of k, e it is enough to show:

$$\mathrm{N}\big((X(s) \cap H)^{\mathrm{tr}}, T\big) \ \leqslant \ c_1 T^{\varepsilon/2}, \text{ for all } T,$$

where $c_1 \in \mathbb{R}^{>}$ depends only on X, ε, not on s, H, T. Below we provide such c_1.

With the present values of e and n, set $D := \binom{e+n}{n}$, $F := \mathbb{R}^D \setminus \{0\}$, and let $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ be as in Lemma 2.3. For $l = 1, \dots, L$, take $\boldsymbol{i}^l = (i^l_1, \dots, i^l_n)$ in $\{0,1\}^n$, not equal to $(1, \dots, 1)$, such that for all $t \in F$ the subset $\mathcal{C}_l(t)$ of $\mathbb{R}^n$ is a semialgebraic $\boldsymbol{i}^l$-cell or empty, so

$$p_{\boldsymbol{i}^l} : \mathbb{R}^n \to \mathbb{R}^{n_l}, \quad n_l := i^l_1 + \dots + i^l_n < n,$$

maps $\mathcal{C}_l(t)$ homeomorphically onto its image. Then we have for $l = 1, \dots, L$ a definable set $Y_l \subseteq (E \times F) \times \mathbb{R}^{n_l}$ such that for all $(s,t) \in E \times F$,

$$Y_l(s,t) \ = \ p_{\boldsymbol{i}^l}\big(X(s) \cap \mathcal{C}_l(t)\big).$$

Since all $n_l < n$ we can assume inductively that for all $(s,t) \in E \times F$ and all T,

$$\mathrm{N}\big(Y_l(s,t)^{\mathrm{tr}}, T\big) \ \leqslant \ B_l T^{\varepsilon/2}, \quad l = 1, \dots, L$$

with $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^>$ independent of s, t, T. Since $H = \mathcal{H}(t)$ for some $t \in F$,

$$\mathrm{N}\big((X(s) \cap H)^{\mathrm{tr}}, T\big) \;\leqslant\; (B_1 + \cdots + B_L)T^{\varepsilon/2},$$

as in the sketch. Thus $c_1 := B_1 + \cdots + B_L$ is as promised. $\square$

Next a variant of Theorem 2.4 where we remove from the sets $X(s)$ only a definable part $V(s)$ of $X(s)^{\mathrm{alg}}$ instead of all of it. The example preceding the statement of Theorem 1.1 shows that this variant is strictly stronger than Theorem 2.4.

Theorem 2.5. *Let any ε be given. Then there is a definable set $V = V(X, \varepsilon) \subseteq X$ and a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T,*

$$V(s) \subseteq X(s)^{\mathrm{alg}} \quad \textit{and} \quad \mathrm{N}\big(X(s) \setminus V(s), T\big) \leqslant cT^{\varepsilon}.$$

Proof. By induction on n. We follow closely the proof of Theorem 2.4. Let $V_0 \subseteq X$ be given by $V_0(s) =$ interior of $X(s)$ in $\mathbb{R}^n$ for $s \in E$. This definable set V_0 will be part of a V as required. Replacing X by $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. We arrange in addition that $X(s) \subseteq [-1, 1]^n$ for all $s \in E$. Now take e and $k = k(n, e)$ as in the proof of Theorem 2.4. It will be enough to find a definable $V \subseteq X$ and a constant $c_1 \in \mathbb{R}^>$ such that for all $s \in E$, every hypersurface H of degree $\leqslant e$ in $\mathbb{R}^n$, and all T we have

$$V(s) \;\subseteq\; X(s)^{\mathrm{alg}}, \qquad \mathrm{N}\big((X(s) \cap H) \setminus V(s), T\big) \;\leqslant\; c_1 T^{\varepsilon/2}.$$

We take the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \ldots, \mathcal{C}_L \subseteq F \times \mathbb{R}^n$ and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$ for $l = 1, \ldots, L$ as in the proof of Theorem 2.4. For such l we have $n_l < n$, so we can assume inductively that we have a definable set $W_l \subseteq Y_l$ and a number $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}^>$ such that for all $s \in E$, $t \in F$, and T we have

$$W_l(s,t) \;\subseteq\; Y_l(s,t)^{\mathrm{alg}} \quad \text{and} \quad N\big(Y_l(s,t) \setminus W_l(s,t), T\big) \;\leqslant\; B_l T^{\varepsilon/2}.$$

It is now easy to check that the definable set $V \subseteq X$ such that for all $s \in E$,

$$V(s) \;=\; \bigcup_{l=1}^{L} \bigcup_{t \in F} \mathcal{C}_l(t) \cap p_{\boldsymbol{i}^l}^{-1}\big(W_l(s,t)\big)$$

has the desired property. $\square$

In the next sections we establish the results used in the proofs above, namely Theorems 1.2 and 1.3. In Section 8 we strengthen and extend Theorem 2.5 in several ways without changing the basic inductive set-up of its proof.

3. Proof of Theorem 1.2

We begin with introducing a key determinant. Let $\boldsymbol{k}$ be a field and set

$$D(n,\ e) \;:=\; \binom{e+n}{n} \;=\; \#\{\alpha \in \mathbb{N}^n :\ |\alpha| \;\leqslant\; e\} \in \mathbb{N}^{\geqslant 1},$$

the dimension of the $\boldsymbol{k}$-linear space of n-variable polynomials over $\boldsymbol{k}$ of (total) degree at most e. Thus $D(n,\ 0) = 1$, $D(n, e) = \frac{e^n}{n!}\big(1 + o(1)\big)$ as $e \to \infty$, and if $n \geqslant 1$, then $D(n, e)$ is strictly increasing as a function of e.

For now we fix n and e, set $D := D(n, e)$ and let α range over $\mathbb{N}^n$. By a *hypersurface in $\boldsymbol{k}^n$ of degree $\leqslant e$* we mean the set of zeros in $\boldsymbol{k}^n$ of a nonzero n-variable polynomial of degree $\leqslant e$ with coefficients in $\boldsymbol{k}$.

Lemma 3.1. *A set $S \subseteq \boldsymbol{k}^n$ is contained in some hypersurface in $\boldsymbol{k}^n$ of degree at most e if and only if* $\det(a_i^\alpha)_{|\alpha|\leqslant e, i=1,\dots,D} = 0$ *for all $a_1,\dots,a_D \in S$.*

Proof. Let $f = \sum_{|\alpha|\leqslant e} c_\alpha x^\alpha$ be a nonzero polynomial in $x = (x_1,\dots,x_n)$ of degree at most e with coefficients $c_\alpha \in \boldsymbol{k}$ such that $f = 0$ on S. Then for any points $a_1,\dots,a_D \in S$ we have $f(a_1) = \cdots = f(a_D) = 0$, that is,

$$\sum_{|\alpha|\leqslant e} c_\alpha\big(a_1^\alpha,\dots,a_D^\alpha\big) \ = \ 0 \ \text{ in } \ \boldsymbol{k}^D,$$

so the D vectors $(a_1^\alpha,\dots,a_D^\alpha)$ ($|\alpha| \leqslant e$) in the $\boldsymbol{k}$-linear space $\boldsymbol{k}^D$ are linearly dependent, which gives the desired conclusion about the determinant.

Conversely, suppose $\det(a_i^\alpha)_{|\alpha|\leqslant e, i=1,\dots,D} = 0$ for all $a_1,\dots,a_D \in S$. Then for $A := \{\alpha : |\alpha| \leqslant e\}$, the $\boldsymbol{k}$-linear subspace of $\boldsymbol{k}^A$ spanned by the vectors $(a^\alpha)_{|\alpha|\leqslant e}$ with $a \in S$ has dimension $< D$. Take $a_1,\dots,a_M \in S$ such that

$$(a_1^\alpha)_{|\alpha|\leqslant e},\ \dots,\ (a_M^\alpha)_{|\alpha|\leqslant e}$$

is a basis of this subspace. Then $M < D$, so we have $c_\alpha \in \boldsymbol{k}$ for $|\alpha| \leqslant e$, with $c_\alpha \neq 0$ for some α and $\sum_{|\alpha|\leqslant e} c_\alpha a^\alpha = 0$ for $a = a_1,\dots,a_M$, and thus for all $a \in S$. $\square$

Next we introduce some numbers related to $D = D(n,e)$:

$$E(n,\ e) \ := \ \binom{e+n-1}{n-1} \ = \ \#\{\alpha :\ |\alpha| = e\},$$

the dimension of the $\boldsymbol{k}$-linear space of homogeneous n-variable polynomials of degree e over $\boldsymbol{k}$. (Here $\binom{-1}{-1} := 1$ and $\binom{k}{-1} := 0$.) So $D(n,e) = \sum_{i=0}^e E(n,i)$. Next, we set $V(n,e) := \sum_{i=0}^e iE(n,i)$. Now for $i \geqslant 1$,

$$iE(n,i) \ = \ i\binom{i+n-1}{n-1} \ = \ n\binom{i+n-1}{n} \ = \ nE(n+1,i-1), \quad \text{so}$$

$$V(n,e) \ = \ n\sum_{i=1}^e E(n+1,i-1) \ = \ nD(n+1,e-1) \text{ for } e \geqslant 1, \quad V(n,0) = 0,$$

and thus for fixed n we have $V(n,e) = \frac{ne^{n+1}}{(n+1)!}\big(1+o(1)\big)$ as $e \to \infty$.

Let $e,m,n \geqslant 1$ below and define $b = b(m,\ n,\ e) \in \mathbb{N}$ by requiring

$$D(m,\ b) \ \leqslant \ D(n,\ e) \ < \ D(m,\ b+1)\ .$$

Next, we set for $b = b(m,n,e)$:

$$\begin{aligned} B(m,\ n,\ e) \ &:= \ \sum_{i=0}^b iE(m,i) + (b+1)\cdot\Big(D(n,\ e) - \sum_{i=0}^b E(m,i)\Big) \\ &= \ V(m,b) + (b+1)\cdot\big(D(n,e) - D(m,b)\big) \in \mathbb{N}^{\geqslant 1}, \\ \varepsilon(m,\ n,\ e) \ &:= \ \frac{mneD(n,e)}{B(m,n,e)}. \end{aligned}$$

Lemma 3.2. *With fixed $m,n \geqslant 1$ and $e \to \infty$, we have:*

1. $b(m,\ n,\ e) \ = \ \big(\frac{m!e^n}{n!}\big)^{1/m}\big(1+o(1)\big)$*;*
2. $B(m,\ n,\ e) \ = \ \frac{m}{(m+1)!}\big(\frac{m!}{n!}\big)^{(m+1)/m} e^{n(m+1)/m}\big(1+o(1)\big)$*;*
3. *if $m < n$, then $\varepsilon(m,\ n,\ e) \to 0$.*

Proof. As to (1), for $e \to \infty$ we have $b = b(m,n,e) \to \infty$, so

$$D(m,b) \;=\; \frac{b^m}{m!}\bigl(1+o(1)\bigr) \;\leqslant\; \frac{e^n}{n!}\bigl(1+o(1)\bigr) \;\leqslant\; \frac{(b+1)^m}{m!}\bigl(1+o(1)\bigr),$$

but the last term here is also $\frac{b^m}{m!}\bigl(1+o(1)\bigr)$, like the first term, and this easily yields the asymptotics claimed for b. For (2), substituting the result of (1) in the asymptotics for $D(m,b)$ as $b \to \infty$ leads to $(b+1)\cdot\bigl(D(n,e)-D(m,b)\bigr) = o\bigl(e^{n(m+1)/m}\bigr)$, and then in the asymptotics for $V(m,b)$ yields the asymptotics claimed for $B(m,n,e)$. Now (3) is an easy consequence of (2). $\square$

In the proof of Proposition 3.4 below we need a reasonable bound on the absolute value of the determinant of a certain $(D\times D)$-matrix of the form $\bigl(a_i^{\alpha}\bigr)_{|\alpha|\leqslant e,\, i=1,\dots,D}$. We achieve this by expressing the matrix as a sum of simpler matrices. In this connection we need a useful expression for the determinant of a sum of matrices.

Turning to this, let $N \in \mathbb{N}$ and consider an $(N\times N)$-matrix $a = (a_{\mu\nu})_{1\leqslant \mu,\nu\leqslant N}$ over a field $\boldsymbol{k}$. The determinant of an $(N\times N)$-matrix over $\boldsymbol{k}$ is an alternating multilinear function of its columns. The columns of a are $a_1,\dots,a_N \in \boldsymbol{k}^N$ where $a_\nu = (a_{1\nu},\dots,a_{N\nu})^t \in \boldsymbol{k}^N$ is the νth column of a. Thus

$$a \;=\; (a_1,\dots,a_N) \in \boldsymbol{k}^N \times \cdots \times \boldsymbol{k}^N \text{ (with } N \text{ factors } \boldsymbol{k}^N).$$

Next, let $a = a^1 + \cdots + a^r$ with $r \in \mathbb{N}$ and $a^1,\dots,a^r$ also $(N\times N)$-matrices over $\boldsymbol{k}$, with a^j having ν^{th} column a^j_ν. Then

$$\begin{aligned}\det a \;&=\; \det\bigl(a_1,\dots,a_N\bigr) \;=\; \det\Bigl(\sum_{j=1}^{r} a_1^j,\dots,\sum_{j=1}^{r} a_N^j\Bigr)\\ &= \sum_{\boldsymbol{j}} \det\bigl(a_1^{j_1},\dots,a_N^{j_N}\bigr)\end{aligned}$$

where $\boldsymbol{j} = (j_1,\dots,j_N)$ ranges here and below over elements of $\{1,\dots,r\}^N$. Let $\boldsymbol{j}$ be given. If for some j in $\{1,\dots,r\}$ the number of $\nu \in \{1,\dots,N\}$ with $j_\nu = j$ is more than $\operatorname{rank} a^j$, then the column vectors $a_1^{j_1},\dots,a_N^{j_N}$ are $\boldsymbol{k}$-linearly dependent, so $\det\bigl(a_1^{j_1},\dots,a_N^{j_N}\bigr) = 0$. Thus if $J \subseteq \{1,\dots,r\}^N$ contains all $\boldsymbol{j}$ such that

$$\#\{\nu \in \{1,\dots,N\} :\; j_\nu = j\} \;\leqslant\; \operatorname{rank} a^j, \text{ for } j = 1,\dots,r,$$

then

$$(*) \qquad \det a \;=\; \sum_{\boldsymbol{j}\in J} \det\bigl(a_1^{j_1},\dots,a_N^{J_N}\bigr) \;=\; \sum_{\boldsymbol{j}\in J} \det\bigl(a_{\mu\nu}^{j_\nu}\bigr)_{1\leqslant\mu,\nu\leqslant N}.$$

We shall also use the following observation:

Lemma 3.3. *Let A be a set and V a finite-dimensional subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^A$. Then for any $N \in \mathbb{N}$, functions $f_1,\dots,f_N \in V$, and points $a_1,\dots,a_N$ in A, the rank of the $(N\times N)$-matrix $\bigl(f_\mu(a_\nu)\bigr)_{1\leqslant\mu,\nu\leqslant N}$ over $\boldsymbol{k}$ is $\leqslant \dim V$.*

Proof. The map $f \mapsto \bigl(f(a_1),\dots,f(a_N)\bigr) : V \to \boldsymbol{k}^N$ is $\boldsymbol{k}$-linear, so the image of this map is a subspace of the $\boldsymbol{k}$-linear space $\boldsymbol{k}^N$ of dimension $\leqslant \dim V$. $\square$

Recall our norm $|(t_1,\dots,t_m)| := \max\{|t_1|,\dots,|t_m|\}$ on $\mathbb{R}^m$ for $m \geqslant 1$.

Proposition 3.4. *Let $e, m, n \geqslant 1$, $m < n$, and $k := b(m, n, e) + 1$. Then there is a constant $K = K(m, n, e)$ with the following property: if $f : (0,1)^m \to \mathbb{R}^n$ is a strong k-parametrization, $0 < r \leqslant 1$, and $a_0, \dots, a_D \in (0,1)^m$ with $D = D(n, e)$ are such that $|a_i - a_0| \leqslant r$ for $i = 1, \dots, D$, then*

$$|\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e,\ i=1,\dots,D}| \ < \ K r^{B(m,n,e)}.$$

Proof. Let $f = (f_1, \dots, f_n)$ with $f_j : (0,1)^m \to \mathbb{R}$. Taylor expansion around a_0 gives for $i = 1, \dots, D$ and $j = 1, \dots, n$, and with $b := b(m, n, e)$:

$$f_j(a_i) \ = \ P_j(a_i - a_0) + R_{ij}(a_i - a_0)$$

where $P_j \in \mathbb{R}[x_1, \dots, x_m]$ has degree $\leqslant b$, the remainder is given by a homogeneous polynomial $R_{ij} \in \mathbb{R}[x_1, \dots, x_m]$ of degree $k = b + 1$, and all coefficients of P_j and R_{ij} are bounded in absolute value by 1. Let $|\alpha| \leqslant e$. Then for $i = 1, \dots, D$ we have

$$\prod_{j=1}^{n} (P_j + R_{ij})^{\alpha_j} \ = \ P_\alpha + R_{i\alpha}$$

with $P_\alpha \in \mathbb{R}[x_1, \dots, x_m]$ of degree $\leqslant b$, the remainder $R_{i\alpha} \in \mathbb{R}[x_1, \dots, x_m]$ has only monomials of degree $> b$, and every coefficient of P_α and $R_{i\alpha}$ is bounded in absolute value by $D(m,k)^{|\alpha|}$, the latter because $\prod_{j=1}^{n}(P_j + R_{ij})^{\alpha_j}$ is a product of $|\alpha|$ factors of the form $\sum c_\beta x^\beta$, with the summation over the $\beta \in \mathbb{N}^m$ with $|\beta| \leqslant k$, and real coefficients c_β with $|c_\beta| \leqslant 1$. Hence for $i = 1, \dots, D$,

$$f(a_i)^\alpha \ = \ \prod_{j=1}^{n} f_j(a_i)^{\alpha_j} \ = \ P_\alpha(a_i - a_0) + R_{i\alpha}(a_i - a_0).$$

We have $D(m,k)^{|\alpha|} \leqslant D(m,k)^e \leqslant c$ for a positive constant $c = c(m, n, e)$ depending only on m, n, e. Next, $P_\alpha = \sum_{j=0}^{b} P_\alpha^j$ where $P_\alpha^j \in \mathbb{R}[x_1, \dots, x_m]$ is homogeneous of degree j. In the matrix algebra $\mathbb{R}^{D\times D}$ this yields the sum decomposition

$$\begin{aligned}\big(f(a_i)^\alpha\big)_{\alpha,i} \ &= \ \sum_{j=0}^{b} \big(P_\alpha^j(a_i - a_0)\big)_{\alpha,i} + \big(R_{i\alpha}(a_i - a_0)\big)_{\alpha,i} \\ &= \ \sum_{j=0}^{k} \big(P_{i\alpha}^j(a_i - a_0)\big)_{\alpha,i}\end{aligned}$$

where $P_{i\alpha}^j := P_\alpha^j$ for $j = 0, \dots, b$ and $P_{i\alpha}^k := R_{i\alpha}$. For $j = 0, \dots, b$ the rank of the matrix $\big(P_{i\alpha}^j(a_i - a_0)\big)_{\alpha,i} = \big(P_\alpha^j(a_i - a_0)\big)_{\alpha,i}$ is at most $E(m,j)$ by Lemma 3.3, so expression $(*)$ for the determinant of such a sum gives

$$\det\big(f(a_i)^\alpha\big)_{\alpha,i} \ = \ \sum_{\boldsymbol{j}\in J} \det\big(P_{i\alpha}^{j_i}(a_i - a_0)\big)_{\alpha,i}$$

where J is the set of all $\boldsymbol{j} = (j_1, \dots, j_D) \in \{0, \dots, b+1\}^D$ such that

$$\#\{\nu \in \{1, \dots, D\} : \ j_\nu = j\} \ \leqslant \ E(m,j), \qquad \text{for } j = 0, \dots, b.$$

Then for $\boldsymbol{j} \in J$ we have $|\det\big(P_{i\alpha}^{j_i}(a_i - a_0)\big)_{\alpha,i}| \leqslant D!c^D r^{|\boldsymbol{j}|}$. It remains to show that for $\boldsymbol{j} \in J$ we have $|\boldsymbol{j}| \geqslant B(m, n, e)$, because then

$$|\det\big(f(a_i)^\alpha\big)_{|\alpha|\leqslant e,\ i=1,\dots,D}| \ \leqslant \ \#J \cdot D!c^D r^{B(m,n,e)},$$

which gives a constant $K = K(m, n, e)$ as claimed.

Fix any $\boldsymbol{j} \in J$, and let $d_j \in \mathbb{N}$ for $j = 0, \dots, b$ be such that

$$\#\{\nu \in \{1, \dots, D\} : \ j_\nu = j\} \ = \ E(m, j) - d_j,$$

and set $N := \#\{\nu \in \{1, \dots, D\} : \ j_\nu = b + 1\}$. Then

$$D \ = \ D(n, e) \ = \ \sum_{j=0}^{b} (E(m, j) - d_j) + N \ = \ D(m, b) - \sum_{j=0}^{b} d_j + N,$$

so $N \ = \ D(n, e) - D(m, b) + d$ with $d := \sum_{j=0}^{b} d_j$. Hence

$$\begin{aligned}
|\boldsymbol{j}| \ &= \ \sum_{\nu=1}^{D} j_\nu \ = \ \sum_{j=0}^{b} j\big(E(m, j) - d_j\big) + (b + 1)N \\
&= \ V(m, b) - \sum_{j=0}^{b} j d_j + (b + 1)\big(D(n, e) - D(m, b) + d\big) \\
&= \ V(m, b) + (b + 1)\big(D(n, e) - D(m, b)\big) + \sum_{j=0}^{b} (b + 1 - j) d_j \\
&= \ B(m, n, e) + \sum_{j=0}^{b} (b + 1 - j) d_j \ \geqslant \ B(m, n, e).
\end{aligned}$$

□

We need one more simple observation:

Lemma 3.5. *Let points $b_1, \dots, b_D \in \mathbb{Q}^n$ with $D = D(n, e)$ be given such that $\mathrm{H}(b_1), \dots, \mathrm{H}(b_D) \leqslant t$, where $t \geqslant 1$. Then*

$$\det \big(b_i^\alpha\big)_{|\alpha| \leqslant e, i} \ \in \ \frac{\mathbb{Z}}{s} \qquad \textit{with } s \in \mathbb{N}^{\geqslant 1}, \ s \leqslant t^{neD}.$$

Proof. For $i = 1, \dots, D$ we have $b_i = (b_{i1}, \dots, b_{in})$ with $b_{ij} = c_{ij}/s_{ij}$, $c_{ij}, s_{ij} \in \mathbb{Z}$, $1 \leqslant s_{ij} \leqslant t$, so

$$b_i^\alpha = \prod_{j=1}^{n} c_{ij}^{\alpha_j} / \prod_{j=1}^{n} s_{ij}^{\alpha_j} \ \in \ \frac{\mathbb{Z}}{s_{i\alpha}}, \quad s_{i\alpha} \ := \ \prod_{j=1}^{n} s_{ij}^{\alpha_j}.$$

Let $\{\alpha : \ |\alpha| \leqslant e\} = \{\alpha_1, \dots, \alpha_D\}$. Then $\det \big(b_i^\alpha\big)_{|\alpha| \leqslant e, i}$ is a sum of terms of the form $\pm \prod_{i=1}^{D} b_i^{\alpha_{\sigma(i)}}$ where σ is a permutation of $\{1, \dots, D\}$. Now the term $\pm \prod_{i=1}^{D} b_i^{\alpha_{\sigma(i)}}$ corresponding to σ lies in $\frac{\mathbb{Z}}{s_\sigma}$ with

$$s_\sigma \ := \ \prod_{i=1}^{D} s_{i\alpha_{\sigma(i)}} \ = \ \prod_{i=1}^{D} \prod_{j=1}^{n} s_{ij}^{\alpha_{\sigma(i)j}}$$

and clearly $s := \prod_{i=1}^{D} \prod_{j=1}^{n} s_{ij}^{e}$ is a common integer multiple of the integers s_σ with $1 \leqslant s \leqslant t^{neD}$, so s has the desired property. □

The following is Theorem 1.2 with more explicit values of k and ε.

Theorem 3.6. *Let $e, m, n \geqslant 1$, $m < n$; set $k := b(m, n, e) + 1$, $\varepsilon := \varepsilon(m, n, e)$. Let $X \subseteq \mathbb{R}^n$ have a strong k-parametrization $f : (0, 1)^m \to \mathbb{R}^n$. Then for all T at most cT^ε hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover $X(\mathbb{Q}, T)$, where $c = c(m, n, e)$ depends only on m, n, e.*

Proof. Let $K = K(m,n,e)$ be as in Proposition 3.4, and let T be given. With $D = D(n,e)$, let $a_1, \dots, a_D \in (0,1)^m$ be such that $f(a_1), \dots, f(a_D) \in X(\mathbb{Q}, T)$. Then Lemma 3.5 gives $s \in \mathbb{N}^{\geqslant 1}$ with $s \leqslant T^{neD}$ (so $T^{-neD} \leqslant 1/s$) such that

$$\det \big(f(a_i)^\alpha\big)_{|\alpha| \leqslant e, i=1,\dots D} \ \in \ \frac{\mathbb{Z}}{s}.$$

Assume also that $0 < r \leqslant 1$ and $a_0 \in (0,1)^m$ are such that $|a_i - a_0| \leqslant r$ for $i = 1, \dots, D$. Can we guarantee that $\det \big(f(a_i)^\alpha\big)_{|\alpha| \leqslant e, i=1,\dots D} = 0$ if r is small enough? Proposition 3.4 gives

$$|\det \big(f(a_i)^\alpha\big)_{|\alpha| \leqslant e,\ i=1,\dots,D}| \ < \ Kr^B, \quad B \ = \ B(m,n,e).$$

So the answer to the question is *yes*: it is enough that $Kr^B \leqslant T^{-neD}$, that is, $r \leqslant \big(K^{-1}T^{-neD}\big)^{1/B}$. Next, considering closed balls of radius r with respect to the norm $|\cdot|$, centered at a point in $(0,1)^m$, how many are enough to cover $(0,1)^m$? For $m = 1$, the interval $(0,1)$ is covered by e segments $[a-r, a+r]$ with $0 < a < 1$, for any natural number e with $2re \geqslant 1$, and there is clearly such an e with $e \leqslant r^{-1}$. Hence at most r^{-m} closed balls of radius r centered at points in $(0,1)^m$ are enough to cover $(0,1)^m$. Taking $r = \big(K^{-1}T^{-neD}\big)^{1/B}$ it follows from Lemma 3.1 that at most $r^{-m} = K^{m/B}T^{mneD/B} = K^{m/B}T^{\varepsilon}$ hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $X(\mathbb{Q}, T)$. So the theorem holds with $c = K^{m/B}$. $\square$

4. Parametrization

Throughout R is an o-minimal field. We refer to part A of the Appendix for the basic facts about o-minimal fields. We rely in particular on the later subsections in this part A concerning differentiation and smoothness. As usual we identify $\mathbb{Q}$ with the prime subfield of R. We drop the subscript R in expressions like $(0,1)_R$ $(= \{t \in R : 0 < t < 1\})$ and $[a,b]_R$ $(= \{t \in R :\ a \leqslant t \leqslant b\})$ for $a < b$ in R.

Let $X \subseteq R^m$ be definable. Call X *strongly bounded* if $X \subseteq [-N, N]^m$ for some N in $\mathbb{N}$. Call a definable map $f : X \to R^n$ strongly bounded if its graph $\Gamma(f) \subseteq R^{m+n}$ is strongly bounded; equivalently, $X \subseteq R^m$ and $f(X) \subseteq R^n$ are strongly bounded.

A *partial k-parametrization of* X is a definable C^k-map $f : (0,1)^l \to R^m$ such that $l = \dim X$, the image of f is contained in X, and $f^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$. A *k-parametrization of* X is a finite set of partial k-parametrizations of X whose images cover X; note that then X is strongly bounded. As a trivial example, if X is finite and strongly bounded, then X has the k-parametrization $\{\phi_a :\ a \in X\}$, where $\phi_a : (0,1)^0 \to R^m$ takes the value a.

The basic ideas for the proofs of the next two parametrization theorems stem from Yomdin [7] and Gromov [2] who considered the semialgebraic case. For our purpose we need to work in an arbitrary o-minimal field.

Theorem 4.1. *Any strongly bounded definable set $X \subseteq R^m$ has for every $k \geqslant 1$ a k-parametrization.*

The inductive proof of this theorem also requires a version for definable maps. A *k-reparametrization* of a definable map $f : X \to R^n$ is a k-parametrization Φ of its domain X such that for every $\phi : (0,1)^l \to R^m$ in Φ, $f \circ \phi$ is of class C^k and $(f \circ \phi)^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leqslant k$; note that then $\{f \circ \phi :\ \phi \in S\}$ is a k-parametrization of $f(X)$, provided $\dim X = \dim f(X)$.

Theorem 4.2. *Any strongly bounded definable map $f : X \to R^n$, $X \subseteq R^m$ has for every $k \geqslant 1$ a k-reparametrization.*

Sections 5,6,7 contain the proof of Theorems 4.1 and 4.2. In Sections 6 and 7 we assume R is $\aleph_0$-saturated, and thus non-archimedean. This can always be arranged by passing to a suitable elementary extension of R and noting that the statements of 4.1 and 4.2 pull back to the original R. (See part B of the Appendix for "$\aleph_0$-saturated" and "elementary extension" and the relevant facts about these notions. See in particular the last two subsections of that part B for a more detailed explanation of how these facts apply to proving Theorems 4.1 and 4.2.)

We often use the following, obtained by repeated use of the Chain Rule:

Lemma 4.3. *Let $f : U \to R$, $g : V \to R$ be definable of class C^k, $k \geqslant 1$, with U, V (definable) open subsets of R. Then $f \circ g : V \cap g^{-1}(U) \to R$ is of class C^k with*

$$(f \circ g)^{(k)} \ = \ \sum_{i=1}^{k} (f^{(i)} \circ g) \cdot p_{ik}\big(g^{(1)}, \dots, g^{(k-i+1)}\big)$$

where the $p_{ik} \in \mathbb{Z}[x_1, \dots, x_{k-i+1}]$ have constant term 0 and $p_{kk} = x_1^k$.

Lemma 4.4. *With $U \subseteq R^l$, $V \subseteq R^m$, let $f : U \to R^m$, $g : V \to R^n$ be definable of class C^k such that $f(U) \subseteq V$ and $f^{(\alpha)}$ and $g^{(\beta)}$ are strongly bounded for all $\alpha \in \mathbb{N}^l$ and $\beta \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and $|\beta| \leqslant k$. Then the definable map $g \circ f : U \to R^n$ is of class C^k with strongly bounded $(g \circ f)^{(\alpha)}$ for all $\alpha \in \mathbb{N}^l$ with $|\alpha| \leqslant k$.*

Some facts about definable families. Recall: $|y| = \max\{|y_1|, \dots, |y_n|\}$ for y in R^n. For a definable map $f : X \to R^n$ with (necessarily definable) $X \subseteq R^m$, we set

$$\|f\| \ := \ \sup_{a \in X} |f(a)| \in [0, +\infty].$$

Note that if X is nonempty and closed and bounded in R^m and f is continuous, then this supremum is a maximum, by Corollary A.8.

Let $m, n \geqslant 1$, $c \in R^{>}$, X a nonempty definable subset of R^m, and $(f_s)_{0<s<1}$ a definable family of maps $f_s : X \to [-c, c]^n$. (That the family is definable simply means that the map $(s, a) \mapsto f_s(a) : (0, 1) \times X \to R^n$ is definable.) Then we have the definable (pointwise) limit map $f_0 : X \to [-c, c]^n$ given by $f_0(a) = \lim_{s \downarrow 0} f_s(a)$. Throughout this subsection s ranges over the elements of R with $0 < s < 1$.

Lemma 4.5. *Suppose the family (f_s) has a Lipschitz constant $\ell \in R^{\geqslant}$, that is, $|f_s(a) - f_s(b)| \leqslant \ell |a - b|$ for all s and all $a, b \in X$; in particular, the f_s are continuous. Then f_0 has Lipschitz constant ℓ, and is thus continuous. If in addition X is closed and bounded in R^m, then $\|f_s - f_0\| \to 0$ as $s \downarrow 0$.*

Proof. Given $a, b \in X$ and taking the limit of $|f_s(a) - f_s(b)|$ as $s \downarrow 0$ we see that f_0 has Lipschitz constant ℓ. Suppose X is closed and bounded. Definable Selection (Proposition A.10) gives a definable 'curve' $\gamma : (0, 1) \to X$ such that $|f_s(\gamma(s)) - f_0(\gamma(s))| = \|f_s - f_0\|$ for all s. Suppose $\|f_s - f_0\|$ does not tend to 0 as $s \downarrow 0$. Then we have $\delta, \varepsilon > 0$ with $\|f_s - f_0\| \geqslant \varepsilon$ for all $s \leqslant \delta$, and thus $|f_s(\gamma(s)) - f_0(\gamma(s))| \geqslant \varepsilon$ for all $s \leqslant \delta$. Now $\gamma(s) \to a \in X$ as $s \downarrow 0$. Then for $s < \delta$,

$$\begin{aligned}
\varepsilon \ &\leqslant \ |f_s(\gamma(s)) - f_0(\gamma(s))| \\
&\leqslant |f_s(\gamma(s)) - f_s(a)| + |f_s(a) - f_0(a)| + |f_0(a) - f_0(\gamma(s))| \\
&\leqslant \ \ell |\gamma(s) - a| + |f_s(a) - f_0(a)| + |f_0(a) - f_0(\gamma(s))|
\end{aligned}$$

but each of the last three terms tends to 0 as $s \downarrow 0$, a contradiction. □

Lemma 4.6. *Suppose X is open in R^m, $k \geqslant 1$, and the maps f_s are of class C^k such that $\|f_s^{(\alpha)}\| \leqslant c$ for all s and all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$. Then $f_0 : X \to R^n$ is of class C^{k-1}, and $f_s^{(\alpha)} \to f_0^{(\alpha)}$ pointwise as $s \downarrow 0$, for all $\alpha \in \mathbb{N}^m$ with $|\alpha| < k$.*

Proof. Let $\alpha \in \mathbb{N}^m$, $|\alpha| < k$, and $a \in X$. Take $\varepsilon > 0$ such that the closed ball

$$B \ := \ [a_1 - \varepsilon, a_1 + \varepsilon] \times \cdots \times [a_m - \varepsilon, a_m + \varepsilon]$$

centered at a with radius ε is contained in X. By MVT (the Mean Value Theorem, Lemma A.13) the definable family $(f_s^{(\alpha)})$ has Lipschitz constant mc on B, so by Lemma 4.5 converges uniformly on B as $s \downarrow 0$ to a continuous definable limit map $B \to [-c, c]^n$; since a is arbitrary, this gives a continuous definable map $f_{0,\alpha} : X \to [-c, c]^n$ such that $f_s^{(\alpha)} \to f_{0,\alpha}$ pointwise as $s \downarrow 0$ (but uniformly on B). Note that $f_{0,\alpha} = f_0$ for $\alpha = (0, \ldots, 0)$. To prove the rest we arrange $n = 1$ by considering the n component functions of f_0 separately; to simplify notation we also assume $m = 1$. (For general m the derivatives are instead appropriate partial derivatives, where only one of the m coordinates varies.) So let $i < k - 1$, and let h range over the elements of R with $|h| \leqslant \varepsilon$ our job is to show that then

$$(*) \qquad \lim_{h \to 0} \frac{f_{0,i}(a+h) - f_{0,i}(a)}{h} \ = \ f_{0,i+1}(a).$$

MVT (Lemma A.11) gives

$$\frac{f_s^{(i)}(a+h) - f_s^{(i)}(a)}{h} \ = \ f_s^{(i+1)}\big(a(s,h)\big)$$

with $a(s, h)$ between a and $a + h$. By Definable Selection (Proposition A.10) we can take $a(s, h)$ definable as a function of (s, h). Then $a(s, h) \to a(h)$ as $s \downarrow 0$ for a definable function $a(h)$ of h. Since $i + 1 < k$ we have by MVT (Lemma A.11):

$$|f_s^{(i+1)}\big(a(s,h)\big) - f_s^{(i+1)}\big(a(h)\big)| \ \leqslant \ c|a(s,h) - a(h)| \ \leqslant \ c|h|.$$

Let $\delta(s) = \max_{b \in B} |f_s^{(i+1)}(b) - f_{0,i+1}(b)|$. Then $\delta(s) \to 0$ as $s \downarrow 0$, by Lemma 4.5, and $f_s^{(i+1)}\big(a(s,h)\big) = f_{0,i+1}\big(a(h)\big) + \delta(s,h)$ with $|\delta(s,h)| \leqslant c|h| + \delta(s)$, and thus

$$\frac{f_s^{(i)}(a+h) - f_s^{(i)}(a)}{h} \ = \ f_{0,i+1}\big(a(h)\big) + \delta(s,h).$$

Fixing h and taking limits as $s \downarrow 0$ we obtain

$$\frac{f_{0,i}(a+h) - f_{0,i}(a)}{h} \ = \ f_{0,i+1}\big(a(h)\big) + \varepsilon(h), \qquad |\varepsilon(h)| \leqslant c|h|.$$

Now $a(h)$ lies between a and $a+h$, endpoints a, $a+h$ included, which gives $(*)$. □

The above analytic facts about definable families properly belong to the topic of function spaces over o-minimal fields, cf. M. Thomas [6].

5. Reparametrizing unary functions

Much in this section is bookkeeping, but we begin with a key analytic fact:

Lemma 5.1. *Let $f : (0,1) \to R$ be a definable C^k-function, $k \geqslant 2$, with strongly bounded $f^{(j)}$ for $0 \leqslant j \leqslant k-1$ and decreasing $|f^{(k)}|$. Define $g : (0,1) \to R$ by $g(t) = f(t^2)$. Then $g^{(j)}$ is strongly bounded for $0 \leqslant j \leqslant k$.*

Proof. Let t range over $(0,1)$. Lemma 4.3 gives

$$g^{(j)}(t) \;=\; \sum_{i=0}^{j} \rho_{ij}(t).f^{(i)}(t^2), \qquad j = 0, \ldots, k$$

where each function ρ_{ij} is given by a 1-variable polynomial with integer coefficients, of degree $\leqslant i$, and with $\rho_{jj}(t) = 2^j t^j$. All summands here are strongly bounded except possibly the one with $i = j = k$, which is $2^k t^k f^{(k)}(t^2)$. So it suffices that $t^k f^{(k)}(t^2)$ is strongly bounded. Let $c \in \mathbb{Q}^{>0}$ be a strong bound for $f^{(k-1)}$. We claim that then $|f^{(k)}(t)| \leqslant 4c/t$ for all t. Suppose towards a contradiction that $t_0 \in (0,1)$ is a counterexample, that is, $|f^{(k)}(t_0)| > 4c/t_0$. Then the Mean Value Theorem (Lemma A.11) provides a $\xi \in [t_0/2, t_0]$ such that

$$f^{(k-1)}(t_0) - f^{(k-1)}(t_0/2) \;=\; f^{(k)}(\xi).(t_0 - t_0/2) \;=\; f^{(k)}(\xi) \cdot t_0/2.$$

Since $|f^{(k)}|$ is decreasing by assumption, $|f^{(k)}(\xi)| \;\geqslant\; |f^{(k)}(t_0)| > 4c/t_0$. Hence

$$2c \;\geqslant\; |f^{(k-1)}(t_0) - f^{(k-1)}(t_0/2)| \;>\; (4c/t_0) \cdot (t_0/2) \;=\; 2c.$$

This contradiction proves our claim. Then for all t,

$$|t^k f^{(k)}(t^2)| \;\leqslant\; t^k \cdot (4c/t^2) \;=\; 4ct^{k-2} \;\leqslant\; 4c$$

using $k \geqslant 2$ for the last inequality. □

The lemma fails for $k = 1$, with $t \mapsto t^{1/3}$ as a counterexample.

Lemma 5.2. *Let $f : (0,1) \to R$ be definable and strongly bounded. Then f has a 1-reparametrization Φ such that for every $\phi \in \Phi$, ϕ or $f \circ \phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. Take elements $a_0 = 0 < a_1 < \cdots < a_n < a_{n+1} = 1$ in R such that, for $i = 0, 1, \ldots, n$, f is of class C^1 on (a_i, a_{i+1}), and either $|f'| \leqslant 1$ on (a_i, a_{i+1}), or $|f'| > 1$ on (a_i, a_{i+1}). Let $i \in \{0, \ldots, n\}$. If $|f'| \leqslant 1$ on (a_i, a_{i+1}), define

$$\phi_i \;:\; (0,1) \to R, \qquad \phi_i(t) \;:=\; a_i + (a_{i+1} - a_i)t.$$

If $|f'| > 1$ on (a_i, a_{i+1}), set

$$b_i \;:=\; \lim_{t \downarrow a_i} f(t), \qquad b_{i+1} \;:=\; \lim_{t \uparrow a_{i+1}} f(t)$$

and as in this case f is continuous and strictly monotone on (a_i, a_{i+1}) we can define $\phi_i : (0,1) \to R$ by $\phi_i(t) = f^{-1}\big(b_i + (b_{i+1} - b_i)t\big)$, where f^{-1} denotes the compositional inverse of the restriction of f to (a_i, a_{i+1}); this compositional inverse has domain (b_i, b_{i+1}) if $b_i < b_{i+1}$, and domain (b_{i+1}, b_i) if $b_i > b_{i+1}$.

In either case, ϕ_i maps $(0,1)$ onto (a_i, a_{i+1}) and both ϕ_i and $f \circ \phi_i$ are of class C^1 with strongly bounded derivative. Moreover, ϕ_i or $f \circ \phi_i$ is given by a univariate polynomial of degree 1 with strongly bounded coefficients in R. Thus

$$\Phi \;:=\; \{\phi_0, \ldots, \phi_n, \widehat{a}_1, \ldots, \widehat{a}_n\}$$

is a 1-reparametrization of f as required, where $\widehat{a}_i$ denotes the constant function on $(0,1)$ with value a_i. $\square$

Lemma 5.3. *Let $k \geqslant 1$ and suppose $f : (0,1) \to R$ is definable and strongly bounded. Then f has a k-reparametrization Φ such that for all $\phi \in \Phi$, ϕ or $f \circ \phi$ is given by a 1-variable polynomial with strongly bounded coefficients in R.*

Proof. By induction on k. The case $k = 1$ is Lemma 5.2. Suppose $k \geqslant 2$ and Φ is a $(k-1)$-reparametrization of f with the additional property. Let $\phi \in \Phi$. Then $\{\phi, f \circ \phi\} = \{g, h\}$ where g is given by a univariate polynomial with strongly bounded coefficients in R. Thus g is of class C^∞, and $g^{(i)}$ is strongly bounded for all $i \in \mathbb{N}$, and h is of class C^{k-1} with strongly bounded $h^{(j)}$ for $j = 0, \dots, k-1$. In order to apply Lemma 5.1 we use o-minimality: take elements

$$a_0 \;=\; 0 \;<\; a_1 \;<\; \dots \;<\; a_{n_\phi} \;<\; a_{n_\phi+1} \;=\; 1$$

in R such that for $i = 0, \dots, n_\phi$, the function h is of class C^k on (a_i, a_{i+1}) and $|h^{(k)}|$ is monotone on (a_i, a_{i+1}). Define $\theta_{\phi,i} : (0,1) \to R$ as $t \mapsto a_i + (a_{i+1} - a_i)t$, if $|h^{(k)}|$ is decreasing, and as $t \mapsto a_{i+1} + (a_i - a_{i+1})t$, otherwise; so $\theta_{\phi,i}$ has image (a_i, a_{i+1}). Then $h \circ \theta_{\phi,i} : (0,1) \to R$ is of class C^k, $(h \circ \theta_{\phi,i})^{(j)}$ is strongly bounded for $j = 0, \dots, k-1$, and $|h \circ \theta_{\phi,i}^{(k)}|$ is decreasing. Let $\rho : (0,1) \to (0,1)$ be the C^∞-bijection sending t to t^2. By Lemma 5.1, the definable C^k-function $h \circ \theta_{\phi,i} \circ \rho : (0,1) \to R$ has strongly bounded jth derivative for $j = 0, \dots, k$. The function $g \circ \theta_{\phi,i} \circ \rho$ is still given by a 1-variable polynomial with strongly bounded coefficients in R, and $\{g \circ \theta_{\phi,i} \circ \rho, h \circ \theta_{\phi,i} \circ \rho\} = \{\phi \circ \theta_{\phi,i} \circ \rho, f \circ (\phi \circ \theta_{\phi,i} \circ \rho)\}$. The images of the functions $\phi \circ \theta_{\phi,i} \circ \rho$ with $i \in \{0, \dots, n_\phi\}$ cover the image of ϕ apart from finitely many points. So adding finitely many constant functions with domain $(0,1)$ and values in $(0,1)$ to the set $\{\phi \circ \theta_{\phi,i} \circ \rho : \ \phi \in S, \ i = 0, \dots, n_\phi\}$ we obtain a k-reparametrization of f as claimed in the statement of the lemma. $\square$

Corollary 5.4. *Let $f : X \to R$ be definable and strongly bounded with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. The case that X is finite is obvious. Suppose X is infinite, $k \geqslant 1$. Since X is a finite union of strongly bounded intervals and points, it has a k-parametrization Φ by univariate polynomial functions of degree $\leqslant 1$. Now Lemma 5.3 provides for every $\phi : (0,1) \to R$ in Φ a k-reparametrization Ψ_ϕ of $f \circ \phi : (0,1) \to R$; then $\{\phi \circ \psi : \ \phi \in \Phi, \ \psi \in \Psi_\phi\}$ is a k-reparametrization of f. $\square$

Next one might reparametrize "curves" $(0,1) \to R^n$ with $n \geqslant 2$, but there is nothing special about the univariate case here, so we do the general case:

Lemma 5.5. *Let $k, m \geqslant 1$, and suppose that every strongly bounded definable function $X \to R$ with $X \subseteq R^l$, $l \leqslant m$, has a k-reparametrization. Then every strongly bounded definable map $X \to R^n$ with $X \subseteq R^l$, $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Proof. Let $n \geqslant 1$, and suppose $F : X \to R^n$ and $f : X \to R$ with $X \subseteq R^m$ are definable, strongly bounded, and F has a k-reparametrization. It is enough to show that then the strongly bounded definable map $(F, f) : X \to R^{n+1}$ has a k-reparametrization. The case of finite X being trivial, assume X is infinite. Let Φ be a k-reparametrization of F and let $\phi \in \Phi$, $\phi : (0,1)^l \to R^m$, $l = \dim X \leqslant m$. Applying the hypothesis of the lemma to the map $f \circ \phi : (0,1)^l \to R$ we obtain a

k-reparametrization Ψ_ϕ of it. Then using Lemma 4.4, $\{\phi \circ \psi : \phi \in \Phi, \ \psi \in \Psi_\phi\}$ is a k-reparametrization of (F, f). □

Remark. At one point we need a slight variant of this lemma, with the same proof: *Let $k, m \geqslant 1$, and suppose that every strongly bounded definable function $(0,1)^l \to R$ with $l \leqslant m$ has a k-reparametrization. Then every strongly bounded definable map $(0,1)^l \to R^n$ with $l \leqslant m$ and $n \geqslant 1$ has a k-reparametrization.*

Corollary 5.6. *Let $n \geqslant 1$ and suppose $f : X \to R^n$ is definable and strongly bounded, with $X \subseteq R$. Then f has a k-reparametrization, for every $k \geqslant 1$.*

Proof. Immediate from Corollary 5.4 and the case $m = 1$ of Lemma 5.5. □

6. Convergence

In this section we assume that our ambient o-minimal field R is $\aleph_0$-saturated.

Let $k, N \in \mathbb{N}^{\geqslant 1}$ and let s, t range over $(0,1)$. Let (F_s) be a definable family of maps

$$F_s \ : \ (0,1) \to (0,1)^N$$

of class C^k with strongly bounded derivatives $F_s^{(i)}$ for $i = 0, \dots, k$. As R is $\aleph_0$-saturated, we have a uniform bound $c \in \mathbb{N}^{\geqslant 1}$ with $|F_s^{(i)}(t)| \leqslant c$ for $i = 0, \dots, k$ and all s, t. Then o-minimality gives a definable limit map

$$F_0 \ : \ (0,1) \to [0,1]^N, \quad F_0(t) \ := \ \lim_{s \downarrow 0} F_s(t),$$

and F_0 is of class C^{k-1}, with $F_0^{(i)}(t) = \lim_{s\downarrow 0} F_s^{(i)}(t)$ for $i = 0, \dots, k-1$, by Lemma 4.6. We have $F_s = (F_{s1}, \dots, F_{sN})$ and set $\Phi_s := \{F_{s1}, \dots, F_{sN}\}$, the set of component functions of F_s. Suppose $\bigcup_{\phi \in \Phi_s} \operatorname{image}(\phi) = (0,1)$ for all s (so Φ_s is a k-parametrization of $(0,1)$ for all s). Now $F_0 = (F_{01}, \dots, F_{0N})$ and we let Φ_0 be the set of functions $\phi|_{\phi^{-1}(0,1)}$ with $\phi \in \{F_{01}, \dots, F_{0N}\}$.

A *cofinite subset of a set* X is a set $X_0 \subseteq X$ such that $X \setminus X_0$ is finite.

Lemma 6.1. *The set Φ_0 has the following properties:*

(A) $\bigcup_{\psi \in \Phi_0} \operatorname{image}(\psi)$ *is a cofinite subset of* $(0,1)$.

(B) *each function $\psi \in \Phi_0$ has as its domain an open subset of $(0,1)$ and is of class C^{k-1} with strongly bounded $\psi^{(i)}$ for $i = 0, \dots, k-1$.*

Proof. Suppose (A) fails. Then o-minimality gives $a < b$ in $(0,1)$ such that $[a,b]$ is disjoint from $\operatorname{image}(\psi)$ for every $\psi \in \Phi_0$ and thus disjoint from $\operatorname{image}(F_{0i})$ for $i = 1, \dots, N$. Let s be given. It follows from $\bigcup_{i=1}^N \operatorname{image}(F_{si}) \supseteq [a,b]$ and o-minimality that for some $i \in \{1, \dots, N\}$, the image of F_{si} contains a segment $[a_s, b_s]$ with $a \leqslant a_s < b_s \leqslant b$ and $b_s - a_s \geqslant (b-a)/(N+1)$. By o-minimality we have a fixed $i \in \{1, \dots, N\}$ and an $\varepsilon \in (0,1)$ such that for all $s < \varepsilon$ the image of F_{si} contains a segment $[a_s, b_s]$ with $a \leqslant a_s < b_s \leqslant b$ and $b_s - a_s \geqslant (b-a)/(N+1)$. Take $\delta \in (0, 1/2)$ so small that $2c\delta < (b-a)/(N+1)$ and let $s < \varepsilon$. Now $|F'_{si}| \leqslant c$, so F_{si} has Lipschitz constant c, and thus the F_{si}-images of the intervals $(0, \delta)$ and $(1-\delta, 1)$ cannot cover a segment $[a_s, b_s]$ as above. Therefore, we have a point $t_s \in [\delta, 1-\delta]$ such that $F_{s,i}(t_s) \in [a,b]$. (We do not need the a_s, b_s any longer.) By Definable Selection (Proposition A.10) we can take t_s as a definable function of $s \in (0, \varepsilon)$. Then for $t_0 := \lim_{s\downarrow 0} t_s$ we have $1 - \delta \leqslant t_0 \leqslant 1 + \delta$. Now for $s < \varepsilon$ we have

$$|F_{0i}(t_0) - F_{si}(t_s)| \ \leqslant \ |F_{0i}(t_0) - F_{0i}(t_s)| + |F_{0i}(t_s) - F_{si}(t_s)|.$$

The first summand on the right tends to 0 as $s \downarrow 0$ because F_{0i} is continuous, and the second does so because $F_{si} \to F_{0i}$ unformly on $[\delta, 1-\delta]$ as $s \downarrow 0$, by Lemma 4.5. Hence $F_{0i}(t_0) \in [a,b]$, contradicting the defining property of $[a,b]$. This finishes the proof of (A). As to (B), just note that F_0 is of class C^{k-1} with $\|F_0^{(i)}\| \leqslant c$ for $i = 0, \dots, k-1$ by Lemma 4.6. □

We now apply this lemma to set up the inductive process for proving Theorems 4.1 and 4.2. For the rest of this section we fix an $m \geqslant 1$.

Some notation and terminology. For definable open $U \subseteq R^{m+1}$, $V \Subset U$ means that V is a definable open subset of R^{m+1} with $V \subseteq U$ and $\dim(U \setminus V) \leqslant m$.

The *normalization* of a definable function $\psi : I \to R$ on an interval $I = (a,b) \subseteq (0,1)$ is the (definable) function $t \mapsto \psi\big((b-a)t + a\big) : (0,1) \to R$; its image is $\psi(I)$.

Notation about "changing the last variable": For $\phi : (0,1) \to R$ we set

$$I_\phi \ : \ (0,1)^{m+1} \to R^{m+1}, \qquad (t_1, \dots, t_m, t_{m+1}) \mapsto \big(t_1, \dots, t_m, \phi(t_{m+1})\big),$$

and for $f : X \to R^n$, $X \subseteq R^{m+1}$ we set

$$f_\phi \ := \ f \circ I_\phi \ : \ (I_\phi)^{-1}(X) \to R^n, \quad (t_1, \dots, t_m, t_{m+1}) \mapsto f\big(t_1, \dots, t_m, \phi(t_{m+1})\big).$$

Lemma 6.2. *Let $k \geqslant 2$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R$ be a strongly bounded definable C^1-function. Suppose also that $\partial f / \partial x_i$ is strongly bounded for $i = 1, \dots, m$. Then there is a $(k-1)$-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi / \partial x_i$ is strongly bounded on V, for $i = 1, \dots, m+1$.*

Proof. We construct Φ from the limit set Φ_0 of a suitable family $(\Phi_s)_{0<s<1}$ as described above. (Lemma 6.1 almost gives that Φ_0 is a $(k-1)$-parametrization.) For $s, t \in (0,1)$, let $U_s(t)$ be the set of those $a \in (0,1)^m$ such that the open ball in R^{m+1} centered at (a,t) with radius s is entirely contained in U; note that $U_s(t) \times \{t\} \subseteq U$, in particular, $U_s(t) \subseteq (0,1)^m$, and $U_s(t)$ is closed in R^m, not just in $(0,1)^m$. Thus for $0 < s, t < 1$ we have a definable continuous function

$$a \mapsto |\frac{\partial f}{\partial x_{m+1}}(a,t)| \ : \ U_s(t) \to R,$$

which achieves its maximum value at some point $a_s(t) \in U_s(t)$, provided $U_s(t)$ is nonempty. By Definable Selection (Proposition A.10) we may take $(s,t) \mapsto a_s(t)$ to be definable, taking by convention the value $(1/2, \dots, 1/2) \in (0,1)^m$ if $U_s(t) = \emptyset$. Then for all $s, t \in (0,1)$ and $a \in U_s(t)$ we have

$$(*) \qquad (a_s(t), t) \in U, \qquad |\frac{\partial f}{\partial x_{m+1}}(a_s(t), t)| \ \geqslant \ |\frac{\partial f}{\partial x_{m+1}}(a,t)|.$$

Now consider the definable family $(g_s)_{0<s<1}$ of maps

$$g_s : (0,1) \to (0,1)^m \times R, \qquad g_s(t) := \big(a_s(t), f(a_s(t), t)\big),$$

where for convenience we set $f\big(a_s(t), t\big) := 0$ if $\big(a_s(t), t\big) \notin U$. By Corollary 5.6 there is for all $s \in (0,1)$ a k-reparametrization of g_s. Now R is $\aleph_0$-saturated, and together with Definable Selection this yields an $N \in \mathbb{N}^{\geqslant 1}$ and a definable family $(F_s)_{0<s<1}$ of maps $F_s : (0,1) \to (0,1)^N$ such that $\Phi_s := \{F_{s1}, \dots, F_{sN}\}$ is a k-reparametrization of g_s for $0 < s < 1$; see end of Appendix for an explanation.

Let Φ_0 be obtained from the family (F_s) as defined just before Lemma 6.1. By partitioning the domains of the functions in Φ_0 and restricting these functions accordingly we can use the Monotonicity Theorem A.1 to obtain a finite collection Φ of functions taking values in $(0,1)$ whose domains are subsets of $(0,1)$ and are either singletons or subintervals of $(0,1)$, and such that any function in Φ whose domain is a subinterval of $(0,1)$ is either constant or strictly monotone. By throwing away the constant functions in Φ (which include those whose domain is a singleton) and replacing each remaining function with its normalization we arrange that Φ is a $(k-1)$-parametrization of a cofinite subset of $(0,1)$. Now set

$$V \ := \ U \cap \bigcap_{\phi\in\Phi} I_\phi^{-1}(U) \ = \ U \setminus \bigcup_{\phi\in\Phi} I_\phi^{-1}[(0,1)^{m+1}\setminus U].$$

The injectivity (and continuity) of the $\phi \in \Phi$ gives $V \Subset U$. For $\phi \in \Phi$ we have $I_\phi(V) \subseteq U$, so the function f_ϕ is of class C^1 on V using $k \geqslant 2$. Let $\phi \in \Phi$; it only remains to show that then $\partial f_\phi/\partial x_i$ is strongly bounded on V for $i = 1,\dots,m+1$. Since R is $\aleph_0$-saturated, it is enough to show, given any point $(a_0,t_0) \in V$, that $\partial f_\phi/\partial x_i$ is strongly bounded just at this point, for $i = 1,\dots,m+1$.

Since $(a_0,\phi(t_0)) \in U$, this is the case for $i = 1,\dots,m$. For the remaining case $i = m+1$, note first that we have $c,d \in [0,1]$ with $c \neq 0$ and a function $\psi \in \Phi_0$ such that $ct+d \in \mathrm{domain}(\psi)$ and $\phi(t) = \psi(ct+d)$ for all $t \in (0,1)$. So it is enough to show for t_1 in the domain of ψ with $\big(a_0,\psi(t_1)\big) \in U$ that $\psi'(t_1)\cdot(\partial f/\partial x_{m+1})\big(a_0,\psi(t_1)\big)$ is strongly bounded. Let such a t_1 be given. By definition of Φ_0 we have a definable family $(\phi_s)_{0<s<1}$ of functions $\phi_s \in \Phi_s$ such that $\lim_{s\downarrow 0}\phi_s(t_1) = \psi(t_1)$ and, as $k \geqslant 2$, $\lim_{s\downarrow 0}\phi_s'(t_1) = \psi'(t_1)$. Hence for all small enough $s \in (0,1)$:

(i) $\big(a_0,\phi_s(t_1)\big) \in U$, $|(\partial f/\partial x_{m+1})\big(a_0,\psi(t_1)\big) - (\partial f/\partial x_{m+1})\big(a_0,\phi_s(t_1)\big)| \leqslant 1$, by the continuity of $\partial f/\partial x_{m+1}$ on U;

(ii) $|\phi_s'(t_1)-\psi'(t_1)|\cdot|(\partial f/\partial x_{m+1})(a_0,\psi(t_1))| \leqslant 1$;

(iii) $a_0 \in U_s\big(\phi_s(t_1)\big)$: use that $\big(a_0,\psi(t_1)\big) \in U$, that U is open in R^{m+1}, and that $\phi_s(t_1) \to \psi(t_1)$ as $s \downarrow 0$.

Take $s \in (0,1)$ such that (i), (ii), (iii) hold. Then

$$\begin{aligned}
|\psi'(t_1)\cdot\frac{\partial f}{\partial x_{m+1}}\big(a_0,\psi(t_1)\big)| \ &\leqslant\ |\phi_s'(t_1)|\cdot|\frac{\partial f}{\partial x_{m+1}}\big(a_0,\psi(t_1)\big)| + 1, \text{ by (ii)},\\
&\leqslant\ |\phi_s'(t_1)|\cdot|\frac{\partial f}{\partial x_{m+1}}\big(a_0,\phi_s(t_1)\big)| + |\phi_s'(t_1)| + 1, \text{ by (i)},\\
&\leqslant\ |\phi_s'(t_1)|\cdot|\frac{\partial f}{\partial x_{m+1}}(b)| + |\phi_s'(t_1)| + 1,
\end{aligned}$$

by (iii) and $(*)$, where $b \ := \ \big(a_s(\phi_s(t_1)),\phi_s(t_1)\big) \in U$.

Now $|\phi_s'(t_1)|$ is strongly bounded, as $\phi_s \in \Phi_s$, so it suffices to show that

$$\phi_s'(t_1)\cdot\frac{\partial f}{\partial x_{m+1}}(b)$$

is strongly bounded. Since Φ_s is a k-reparametrization of g_s, we have:

(iv) $(a_s\circ\phi_s)'(t_1)$ is strongly bounded, and

(v) $(d/dt)|_{t=t_1} f\big(a_s(\phi_s(t)),\phi_s(t)\big)$ is strongly bounded.

By the Chain Rule (see subsection "Differentiability" in part B of the Appendix), the quantity in (v) equals

$$\sum_{i=1}^m (a_{si}\circ\phi_s)'(t_1)\cdot\frac{\partial f}{\partial x_i}(b) \ + \ \phi_s'(t_1)\cdot\frac{\partial f}{\partial x_{m+1}}(b)$$

The left hand sum here is strongly bounded by (iv) and the strong boundedness of the functions $\partial f/\partial x_i$ for $i = 1,\dots,m$. Hence the right hand term is strongly bounded, which we already showed to be enough. $\square$

Corollary 6.3. *Let $k \geqslant 2$, $n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable C^1-map. Suppose also that $\partial f/\partial x_i$ is strongly bounded for $i = 1,\dots,m$. Then there is a $(k-1)$-parametrization Φ of a cofinite subset of $(0,1)$ and a set $V \Subset U$ such that for every $\phi \in \Phi$: $I_\phi(V) \subseteq U$, f_ϕ is of class C^1 on V, and $\partial f_\phi/\partial x_i$ is strongly bounded on V for $i = 1,\dots,m+1$.*

Proof. For $n = 1$ this is Lemma 6.2. As an inductive assumption, let $f : U \to R^n$ be as in the hypothesis of the corollary and Φ and V as in its conclusion. Let $g : U \to R$ be a strongly bounded definable C^1-function such that $\partial g/\partial x_i$ is strongly bounded for $i = 1,\dots,m$. Then the strongly bounded definable C^1-map $(f,g) : U \to R^{n+1}$ has strongly bounded partial $\partial(f,g)/\partial x_i = (\partial f/\partial x_i, \partial g/\partial x_i)$ for $i = 1,\dots,m$. It now suffices to show that there is a $(k-1)$-parametrization Θ of a cofinite subset of $(0,1)$ and a set $W \Subset U$ such that for all $\theta \in \Theta$: $I_\theta(W) \subseteq U$, $(f,g)_\theta$ is of class C^1 on W, and $\partial(f,g)_\theta/\partial x_i$ is strongly bounded on W for $i = 1,\dots,m+1$. To construct Θ and W, let $\phi \in \Phi$. Then applying Lemma 6.2 to the function $g_\phi : V \to R$ gives a $(k-1)$-parametrization Ψ_ϕ of a cofinite subset of $(0,1)$ and a set $V_\phi \Subset V$ such that for all $\psi \in \Psi_\phi$: $I_\psi(V_\phi) \subseteq V$ and $(g_\phi)_\psi = g_{\phi\circ\theta}$ is of class C^1 on V_ϕ, and $\partial g_{\phi,\psi}/\partial x_i$ is strongly bounded on V_ϕ. Now we set

$$\Theta := \{\phi \circ \psi : \ \phi \in \Phi,\ \psi \in \Psi_\phi\}, \quad W := \bigcap_{\phi\in\Phi} V_\phi.$$

It follows easily from Lemma 4.4 that Θ and W have the desired properties. $\square$

To state the next corollary, let U be a definable open subset of R^{m+1}. Then we have for $t \in R$ the definable open subset U^t of R^m given by

$$U^t \ = \ \{(t_1,\dots,t_m) \in R^m : \ (t_1,\dots,t_m,t) \in U\}.$$

We call a definable map $f : U \to R^n$ *of class C^k in the first m variables* if for every $t \in R$ the (definable) map

$$f^t : U^t \to R^n, \qquad (t_1,\dots,t_m) \mapsto f(t_1,\dots,t_m,t)$$

is of class C^k. In that case $f^{(\alpha)}$ for $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ denotes the definable map

$$(t_1,\dots,t_m,t) \mapsto (f^t)^{(\alpha)}(t_1,\dots,t_m) \ : \ U \to R^n,$$

which for fixed t is continuous as a function of $(t_1,\dots,t_m)$.

Corollary 6.4. *Let $k,n \geqslant 1$, $U \Subset (0,1)^{m+1}$ and let $f : U \to R^n$ be a strongly bounded definable map that is of class C^k in the first m variables, such that $f^{(\alpha)}$ is strongly bounded for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$. Then for every $l \leqslant k$ there is a $V_l \Subset U$ and a k-parametrization Φ_l of a cofinite subset of $(0,1)$ such that for all $\phi \in \Phi_l$: $I_\phi(V_l) \subseteq U$, f_ϕ is of class C^k on V_l and $f_\phi^{(\alpha)} := \big(f_\phi\big)^{(\alpha)}$ is strongly bounded on V_l for all $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l$.*

Proof. The last sentence in the subsection on C^k-maps in part A of the Appendix gives $V_0 \Subset U$ such that f is of class C^k on V_0. Then V_0 and $\Phi_0 = \{\mathrm{id}|_{(0,1)}\}$ have the desired properties for $l = 0$. Suppose, inductively, that $l < k$ and V_l and Φ_l are as stated in the Corollary. Let

$$\Delta \ := \ \{\alpha \in \mathbb{N}^{m+1} \ : \ |\alpha| \leqslant k-1,\ \alpha_{m+1} \leqslant l\},$$

set $\widetilde{n} := \#\Delta \cdot \#\Phi_l$, and let $F_1, \ldots, F_{\widetilde{n}} : V_l \to R^n$ enumerate the set of C^1-maps

$$\{f_\phi^{(\alpha)} : \ V_l \to R^n : \ \alpha \in \Delta, \ \phi \in \Phi_l\}.$$

Then we can apply Corollary 6.3 to $F := (F_1, \ldots, F_{\widetilde{n}}) : V_l \to R^{\widetilde{n} \cdot n}$ in the role of f, and V_l, $\widetilde{n} \cdot n$, $k+1$ instead of U, n, k. This gives a k-parametrization Ψ of a cofinite subset of $(0,1)$ and a set $V_{l+1} \Subset V_l$ such that for all $\psi \in \Psi$: $I_\psi(V_{l+1}) \subseteq V_l$, F_ψ is of class C^1 on V_{l+1}, and $\partial F_\psi / \partial x_i$ is strongly bounded on V_{l+1} for $i = 1, \ldots, m+1$. Next we set

$$\Phi_{l+1} \ := \ \{\phi \circ \psi : \ \phi \in \Phi_l, \ \psi \in \Psi\}.$$

Then Φ_{l+1} is a k-parametrization of a cofinite subset of $(0,1)$ and $I_\theta(V_{l+1}) \subseteq U$, with f_θ of class C^k for all $\theta \in \Phi_{l+1}$.

Let $\theta = \phi \circ \psi$ with $\phi \in \Phi_l, \psi \in \Psi$ and let $\alpha \in \mathbb{N}^{m+1}$, $|\alpha| \leqslant k$, $\alpha_{m+1} \leqslant l+1$; it remains to show that then $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. If $\alpha_{m+1} = 0$, then this holds because $f_\theta^{(\alpha)} = \big(f_\phi^{(\alpha)}\big)_\psi$ and $f_\phi^{(\alpha)}$ is strongly bounded on V_l. Suppose that $\alpha_{m+1} > 0$. Then $\alpha = \beta + (0, \ldots, 0, j)$ with $\beta_{m+1} = 0$ and $j = \alpha_{m+1} \geqslant 1$, so for $a = (a_1, \ldots, a_m, a_{m+1}) \in V_{l+1}$ we have

$$\begin{aligned} f_\theta^{(\alpha)}(a) \ &= \ \frac{\partial^j f_\theta^{(\beta)}}{\partial x_{m+1}^j}(a) \ = \ \frac{\partial^j \big(f_\phi^{(\beta)}\big)_\psi}{\partial x_{m+1}^j}(a) \\ &= \ \sum_{i=1}^{j} \frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}\big(a_1, \ldots, a_m, \psi(a_{m+1})\big) \cdot p_{ij}\big(\psi^{(1)}(a_{m+1}), \ldots, \psi^{(j-i+1)}(a_{m+1})\big) \end{aligned}$$

using Lemma 4.3 and the polynomials p_{ij} from that lemma for the last equality. Since we assumed inductively that the $\frac{\partial^i f_\phi^{(\beta)}}{\partial x_{m+1}^i}$ are strongly bounded on V_l and $\psi^{(1)}, \ldots, \psi^{(k)}$ are strongly bounded on $(0,1)$, $f_\theta^{(\alpha)}$ is strongly bounded on V_{l+1}. $\square$

7. Finishing the proofs of the parametrization theorems

We continue to work in an ambient $\aleph_0$-saturated o-minimal field R. We consider the following statements depending on m:

$(\mathrm{I})_m$ For all $k, n \geqslant 1$, every strongly bounded definable map $f : (0,1)^m \to R^n$ has a k-reparametrization.

$(\mathrm{II})_m$ For all $k \geqslant 1$, every strongly bounded definable set $X \subseteq R^{m+1}$ has a k-parametrization.

It is clear that $(\mathrm{I})_0$ and $(\mathrm{II})_0$ hold; $(\mathrm{I})_1$ holds by Corollary 5.6. We proceed by induction to show that $(\mathrm{I})_m$ and $(\mathrm{II})_m$ hold for all m. So let $m \geqslant 1$ and suppose that $(\mathrm{I})_l$ holds for all $l \leqslant m$ and that $(\mathrm{II})_l$ holds for all $l < m$. We show that then $(\mathrm{II})_m$ holds and next that $(\mathrm{I})_{m+1}$ holds. For $(\mathrm{II})_m$, let $k \geqslant 1$ and let $X \subseteq R^{m+1}$ be definable and strongly bounded. In order to show that X has a k-parametrization we can reduce to the case that X is a cell in R^{m+1}; we do the more difficult of the two cases, namely $X = (f,g)_Y$ where Y is a (strongly bounded) cell in R^m, and $f, g : Y \to R$ are strongly bounded continuous definable functions with $f(y) < g(y)$ for all $y \in Y$; the other case, where X is the graph of such a function $Y \to R$, is left to the reader.

Using $(\mathrm{II})_{m-1}$ we have a k-parametrization Φ of Y. Set $l := \dim Y$. Let $\phi \in \Phi$ be given. Then $\phi : (0,1)^l \to Y$ and $(\mathrm{I})_l$ gives a k-reparametrization Ψ_ϕ of the map

$(f\circ\phi, g\circ\phi):(0,1)^l\to R^2$. For $\psi\in\Psi_\phi$ we have $\psi:(0,1)^l\to(0,1)^l$, and we define $\theta_{\phi,\psi}:(0,1)^{l+1}\to X$ by

$$\theta_{\phi,\psi}(s,t) \ := \ \big((\phi\circ\psi)(s),\ (1-t)\cdot(f\circ\phi\circ\psi)(s)+t\cdot(g\circ\phi\circ\psi)(s)\big)$$

where $(s,t)=(s_1,\dots,s_l,t)\in(0,1)^{l+1}$. Then the set $\{\theta_{\phi,\psi}:\ \phi\in\Phi,\ \psi\in\Psi_\phi\}$ is readily seen to be a k-parametrization of X, and we have established $(\mathrm{II})_m$.

For $(\mathrm{I})_{m+1}$ we need only do the case $n=1$ by the remark following the proof of Lemma 5.5. So let $k\geqslant 1$ and let $f:(0,1)^{m+1}\to R$ be a strongly bounded definable function; our job is to show that f has a k-reparametrization.

In the rest of this proof t ranges over the interval $(0,1)$. By $(\mathrm{I})_m$ there is for all t a k-reparametrization of the function $f^t:(0,1)^m\to R$ given by $f^t(s)=f(s,t)$. Using a saturation and definable selection argument as in the proof of Lemma 6.2 gives an $N\in\mathbb{N}^{\geqslant 1}$ and definable families $(\phi_1^t),\dots,(\phi_N^t)$ of maps

$$\phi_j^t \ : \ (0,1)^m\to(0,1)^m \qquad (j=1,\dots,N)$$

such that $\Phi^t:=\{\phi_1^t,\dots,\phi_N^t\}$ is for every t a k-reparametrization of f^t.

Now, for $j=1,\dots,N$ we define the function $f_j:(0,1)^{m+1}\to R$ by

$$f_j(s,t) \ := \ f\big(\phi_j(s,t),t\big),$$

where $\phi_j:(0,1)^{m+1}\to(0,1)^m$ is given by $\phi_j(s,t):=\phi_j^t(s)$. Consider the map

$$F \ := \ \big(\phi_1,\dots,\phi_N,f_1,\dots,f_N\big) \ : \ (0,1)^{m+1}\to R^{Nm+N}.$$

Then the hypotheses of Corollary 6.4 are satisfied for F and $(0,1)^{m+1}$ in the role of f and U, and $Nm+N$ for n: this is just restating that Φ^t is a k-reparametrization of f^t, uniformly in t. The conclusion of that corollary for $l=k$ gives a set $V\Subset(0,1)^{m+1}$ and a k-parametrization Ψ of a cofinite subset of $(0,1)$ such that for all $\psi\in\Psi$ the map $F_\psi:(0,1)^{m+1}\to R^{Nm+N}$ is of class C^k on V with strongly bounded $F_\psi^{(\alpha)}$ on V for all $\alpha\in\mathbb{N}^{m+1}$ with $|\alpha|\leqslant k$.

For $j=1,\dots,N$ and $\psi\in\Psi$, let $\phi_j*\psi:(0,1)^{m+1}\to(0,1)^{m+1}$ be given by

$$(\phi_j*\psi)(s,t) \ := \ \big(\phi_j(s,\psi(t)),\psi(t)\big) \ = \ \big(\phi_j^{\psi(t)}(s),\psi(t)\big).$$

The images of the $\psi\in\Psi$ cover a set $(0,1)\setminus\{t_1,\dots,t_d\}$ and for every t the images of $\phi_1^t,\dots,\phi_N^t$ cover $(0,1)^m$, and thus the images of the above $\phi_j*\psi$ cover $(0,1)^{m+1}$ apart from finitely many hyperplanes $x_{m+1}=t_i$. Setting

$$W \ := \bigcup_{1\leqslant j\leqslant N,\ \psi\in\Psi}(\phi_j*\psi)(V)$$

it follows that the definable set $(0,1)^{m+1}\setminus W$ has dimension $\leqslant m$. Using the now established $(\mathrm{II})_m$, let Θ_1 be a k-parametrization of V and Θ_2 a k-parametrization of $(0,1)^{m+1}\setminus W$. For $\theta\in\Theta_2$ we have $\theta:(0,1)^l\to(0,1)^{m+1}$ with $l\leqslant m$ and then $(\mathrm{I})_l$ yields a k-reparametrization Λ_θ of the function $f\circ\theta:(0,1)^l\to R$. The required k-reparametrization of f is now given by

$$\{(\phi_j*\psi)\circ\chi:\ j=1,\dots,N,\ \psi\in\Psi,\chi\in\Theta_1\}\cup\{\theta\circ\widehat{\lambda}:\ \theta\in\Theta_2,\lambda\in\Lambda_\theta\}$$

where $\widehat{\lambda}:(0,1)^{m+1}\to(0,1)^l$ (for $l\leqslant m$ as above) is given by $\widehat{\lambda}(t_1,\dots,t_{m+1}):=\lambda(t_1,\dots,t_l)$. This finishes the proof of $(\mathrm{I})_{m+1}$, and the induction is complete. In particular, Theorem 4.1 is now established. Theorem 4.2 requires one more easy step and we leave this to the reader.

Corollary 7.1. *Let $k, n \geqslant 1$; suppose $X \subseteq [-1,1]^n$ is definable, $d := \dim X \geqslant 0$. Then there exists a finite set Φ of definable C^k-maps $f : (0,1)^d \to R^n$ such that*

(i) $\bigcup_{f \in \Phi} \mathrm{image}(f) = X$;
(ii) $|f^{(\alpha)}(t)| \leqslant 1$ *for all* $f \in \Phi$ *and* $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$ *and all* $t \in (0,1)^d$.

Proof. Let Φ^* be a k-parametrization of X. Then (i) holds for Φ^* instead of Φ and (ii) holds for Φ^* instead of Φ, with a certain $c \in \mathbb{N}^{\geqslant 1}$ in place of 1. Cover $(0,1)^d$ with $(c+1)^d$ translates of the 'box' $(0, \frac{1}{c})^d$ and for each such translate B, let $\lambda_B : (0,1)^d \to B$ be the obvious affine bijection. Then the set of maps $f \circ \lambda_B$ as f varies over Φ^* and B over the above translates is the required Φ, since $(f \circ \lambda_B)^{(\alpha)} = c^{-|\alpha|} \cdot \big(f^{(\alpha)} \circ \lambda_B\big)$ for such f and B and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. □

Definable Selection and $\aleph_0$-saturation lead to a uniform version, as explained in more detail at the end of part B of the Appendix:

Corollary 7.2. *Let d, k, m, n be given with $k, n \geqslant 1$ and suppose $E \subseteq R^m$ and*

$$Z \;\subseteq\; E \times [-1,1]^n \;\subseteq\; R^{m+n}$$

are definable with $\dim Z(s) = d$ *for all* $s \in E$. *Then there are* $N \in \mathbb{N}^{\geqslant 1}$ *and a definable set* $F \subseteq E \times R^d \times R^{Nn}$ *such that for all* $s \in E$, $F(s) \subseteq R^d \times R^{Nn}$ *is the graph of a* C^k*-map* $(f_1, \dots, f_N) : (0,1)^d \to (R^n)^N = R^{Nn}$ *such that:*

(i) $\bigcup_{j=1}^N \mathrm{image}(f_j) = Z(s)$;
(ii) $|f_j^{(\alpha)}(t)| \leqslant 1$ *for* $j = 1, \dots, N$, $\alpha \in \mathbb{N}^d$ *with* $|\alpha| \leqslant k$, *and* $t \in (0,1)^d$.

The proof of Corollary 7.2 uses that R is $\aleph_0$-saturated, but this corollary goes through without this assumption: pass to an $\aleph_0$-saturated elementary extension and then go back. Thus it applies to o-minimal expansions of the real field to give Theorem 1.3, and we can also combine it with Theorem 3.6 to give:

Corollary 7.3. *Let $n \geqslant 1$ and let an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field be given. Suppose $E \subseteq \mathbb{R}^m$ and $Z \subseteq E \times [-1,1]^n \subseteq \mathbb{R}^{m+n}$ are definable. Then there is for every $\varepsilon > 0$ an $e = e(\varepsilon, n)$ and a K with the following property: for all $s \in E$ with* $\dim Z(s) < n$ *and all T, at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$.*

The expression "$e = e(\varepsilon, n)$" means: e can be chosen to depend only on ε and n. The proof below uses the numbers $\varepsilon(d, n, e) := \frac{dneD(n,e)}{B(d,n,e)}$ from Section 3.

Proof. Replacing E by finitely many definable subsets over each of which $\dim Z(s)$ takes a given value, we arrange that for a certain $d < n$ we have $\dim Z(s) = d$ for all $s \in E$. If $d = 0$, then we have $K \in \mathbb{N}^{\geqslant 1}$ such that $\#Z(s) \leqslant K$ for all $s \in E$, and so at most K hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant 1$ are enough to cover $Z(s)$. Assume $d \geqslant 1$. Take $e \geqslant 1$ such that $\varepsilon(d, n, e) \leqslant \varepsilon$ and set $k := b(d, n, e) + 1$ as in Theorem 3.6. Corollary 7.2 gives an $N \in N^{\geqslant 1}$ and for every $s \in E$ maps $f_1, \dots, f_N : (0,1)^d \to R^n$ of class C^k such that $Z(s) = \bigcup_{j=1}^N \mathrm{image}(f_j)$ and $|f_j^{(\alpha)}(t)| \leqslant 1$ for $j = 1, \dots, N$ and all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ and all $t \in (0,1)^d$. Applying Theorem 3.6 to each map f_j separately we obtain that for $K := N \cdot C(d, n, e)$ at most KT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$ are enough to cover the set $Z(s)(\mathbb{Q}, T)$. □

8. Strengthening and Extending the Counting Theorem

In this section we fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, and *definable* is with respect to $\widetilde{\mathbb{R}}$. Throughout $n \geqslant 1$ and $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

A closer look at the proof of Theorem 2.5 gives useful extra information about the definable subsets $V(s)$ of $X(s)^{\text{alg}}$: Theorem 8.4. To express this information efficiently requires the notion of a *block family*, which is here simpler than in [4] and well suited to the inductive set-up of Section 2. See the subsection *Dimension* in part A of the Appendix for the local dimension $\dim_a$ used in defining blocks.

A block family version of the Counting Theorem. Let $d \leqslant n$. A *block in* $\mathbb{R}^n$ *of dimension* d is a definable connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$ for which $\dim_a A = d$ for all $a \in A$. Thus the empty subset of $\mathbb{R}^n$ counts as a block in $\mathbb{R}^n$ of dimension d, but if B is a nonempty block in $\mathbb{R}^n$ of dimension d, then $\dim B = d$. Also, a nonempty block of dimension 0 in $\mathbb{R}^n$ consists just of one point. A *block family in* $\mathbb{R}^n$ *of dimension* d is a definable set $V \subseteq E \times \mathbb{R}^n$ all whose sections $V(s)$ are blocks in $\mathbb{R}^n$ of dimension d. Here are two easy lemmas:

Lemma 8.1. *Suppose $U \subseteq \mathbb{R}^m$ is open and semialgebraic, $m \geqslant 1$, and $f : U \to \mathbb{R}^n$ is semialgebraic and maps U homeomorphically onto $f(U)$. Then f maps any block $B \subseteq U$ in $\mathbb{R}^m$ of dimension $d \leqslant m$ onto a block $f(B)$ in $\mathbb{R}^n$ of dimension d.*

In the proof of Theorem 8.4 we apply Lemma 8.1 for every $I \subseteq \{1, \dots, n\}$ to the map $a \mapsto b : \{a \in \mathbb{R}^n : a_i \neq 0 \text{ for } i \in I\} \to \mathbb{R}^n$ with $b_i = a_i^{-1}$ for $i \in I$ and $b_i = a_i$ for $i \notin I$; these maps extend the maps f_I from Section 2.

Lemma 8.2. *Let B be a block in $\mathbb{R}^n$ of dimension $d \leqslant n$. Then B is a union of connected semialgebraic subsets of dimension d.*

Proof. Take semialgebraic $A \subseteq \mathbb{R}^n$ such that $\dim_a A = d$ for all $a \in A$, and B is an open subset of A. For $b \in B$, take a semialgebraic open neighborhood U of b in A such that $U \subseteq B$. Now use that the connected components of U are open in A, by Corollary A.3, and thus of dimension d. $\square$

Corollary 8.3. *Let $Y \subseteq \mathbb{R}^n$ and $1 \leqslant d \leqslant n$.*

(i) *if $B \subseteq Y$ and B is a block in $\mathbb{R}^n$ of dimension d, then $B \subseteq Y^{\text{alg}}$;*
(ii) *if V is a block family in $\mathbb{R}^n$ of dimension d, then the union of the sections of V that are contained in Y is contained in Y^{alg}.*

For the inductive proof below we also define a *block family in* $\mathbb{R}^0$ *of dimension* 0 to be a definable set $V \subseteq E \times \mathbb{R}^0$, with $E \times \mathbb{R}^0$ identified with E in the obvious way.

Theorem 8.4. *Let ε be given. Then there are a natural number $N = N(X, \varepsilon) \geqslant 1$, a block family $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ in $\mathbb{R}^n$ of dimension $d_j \leqslant n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X, \varepsilon)$, such that:*

(i) *$V_j(s,t) \subseteq X(s)$ for $j = 1, \dots, N$ and $(s,t) \in E \times F_j$;*
(ii) *for all T and all $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most cT^{ε} blocks $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.*

This yields an improved Theorem 2.5 as follows. Let $V_1, \dots, V_N$ and c be as in Theorem 8.4. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \ := \bigcup_{d_j \geqslant 1, t \in F_j} V_j(s,t)$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}(X(s)\setminus V(s),T)\leqslant cT^{\varepsilon}$ for all T.

Proof. If Theorem 8.4 holds for definable sets $X_1,\dots,X_\nu\subseteq E\times\mathbb{R}^n$, $\nu\in\mathbb{N}$, then also for $X=X_1\cup\dots\cup X_\nu$. We shall tacitly use this below.

We proceed by induction on n, and follow the proof of Theorem 2.5 closely. Set $V_0(s):=$ interior of $X(s)$. Then [5, (III, 3.6)] gives $M\in\mathbb{N}^{\geqslant 1}$ such that for all $s\in E$,

$$\#\{\text{connected components of } V_0(s)\}\ \leqslant\ M.$$

Definable Selection and the lexicographic ordering on $\mathbb{R}^n$ give definable subsets $V_1,\dots,V_M$ of $E\times\mathbb{R}^n$ such that for all $s\in E$ the sets $V_1(s),\dots,V_M(s)$ are connected (possibly empty), open in $V_0(s)$, pairwise disjoint, with $V(s)\ =\ \bigcup_{i=1}^M V_i(s)$. So $V_1,\dots,V_M$ are block families in $\mathbb{R}^n$ of dimension n; we make them the first M of the $V_1,\dots,V_N$ to be constructed. Now replacing X with $X\setminus V_0$ we arrange that $X(s)$ has empty interior for all $s\in E$. Applying Lemma 8.1 to the natural extensions of the maps f_I, $I\subseteq\{1,\dots,n\}$, we arrange also that $X(s)\subseteq[-1,1]^n$ for all $s\in E$.

Next, take e and $k=k(n,e)$ as in the proof of Theorem 2.4. So we have $C=C(X,\varepsilon)\in\mathbb{R}^{>}$ such that for any $s\in E$, $X(s)(\mathbb{Q},T)$ is covered by at most $CT^{\varepsilon/2}$ many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant e$. Therefore it suffices to find $V_1,\dots,V_N$ and c as in the theorem but with (ii) replaced by

- (ii)* for all T, all $s\in E$, and all hypersurfaces H of degree $\leqslant e$, $(X(s)\cap H)(\mathbb{Q},T)$ is covered by at most $\frac{c}{C}T^{\varepsilon/2}$ blocks $V_j(s,t)$, $(1\leqslant j\leqslant N,\ t\in F_j)$;

We use again the semialgebraic sets $\mathcal{H},\mathcal{C}_1,\dots,\mathcal{C}_L\subseteq F\times\mathbb{R}^n$, and the definable sets $Y_l\subseteq E\times F\times\mathbb{R}^{n_l}$, $l=1,\dots,L$, as in the proof of Theorem 2.4. Since $n_l<n$, the induction assumption gives a natural number $N_l=N(Y_l,\varepsilon)\geqslant 1$, a block family

$$W_{l,i}\ \subseteq\ \big((E\times F)\times G_{l,i}\big)\times\mathbb{R}^{n_l}$$

in $\mathbb{R}^{n_l}$ of dimension $d_{l,i}\leqslant n_l$ with definable $G_{l,i}\subseteq\mathbb{R}^{m_{l,i}}$, for $i=1,\dots,N_l$, and $B_l=B_l(Y_l,\varepsilon)\in\mathbb{R}^{>}$, such that

- (i)$'$ $W_{l,i}(s,t,g)\subseteq Y_l(s,t)$ for $i=1,\dots,N_l$, $(s,t,g)\in(E\times F)\times G_{l,i}$;
- (ii)$'$ for all T and all $(s,t)\in E\times F$, $Y_l(s,t)(\mathbb{Q},T)$ is covered by at most $B_lT^{\varepsilon/2}$ blocks $W_{l,i}(s,t,g)$, $(1\leqslant i\leqslant N_l,\ g\in G_{l,i})$.

Set $N:=N_1+\dots+N_L$, and for $l=1,\dots,L$, $1\leqslant i\leqslant N_l$ and $j=N_1+\dots+N_{l-1}+i$, set $F_j:=F\times G_{l,i}$, and let $V_j\subseteq(E\times F_j)\times\mathbb{R}^n$ be the definable set given by

$$V_j\big(s,(t,g)\big)=\mathcal{C}_l(t)\cap p_{\boldsymbol{i}^l}^{-1}\big(W_{l,i}(s,t,g)\big),\qquad (s\in E,\ t\in F,\ g\in G_{l,i}),$$

so V_j is a block family in $\mathbb{R}^n$ of dimension $d_{l,i}<n$, by Lemma 8.1. It is easy to check that $V_1,\dots,V_N$ and $c:=C(B_1+\dots+B_L)$ are as desired. ☐

A generalization. In this subsection we fix $d\geqslant 1$. Instead of rational points we now allow points with coordinates in a $\mathbb{Q}$-linear subspace of $\mathbb{R}$ of dimension $\leqslant d$. Let $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{R}^d$, and set $\mathbb{Q}\lambda:=\mathbb{Q}\lambda_1+\dots+\mathbb{Q}\lambda_d\subseteq\mathbb{R}$. For $a\in\mathbb{Q}\lambda$ we set

$$\mathrm{H}_\lambda(a)\ :=\ \min\{\mathrm{H}(q):\ q\in\mathbb{Q}^d,\ q\cdot\lambda=a\}\in\mathbb{N}^{\geqslant 1}.$$

Here $q\cdot\lambda:=q_1\lambda_1+\dots+q_d\lambda_d$. We define a height function H_λ on $(\mathbb{Q}\lambda)^n\subseteq\mathbb{R}^n$ by

$$\mathrm{H}_\lambda(a)\ =\ \max\{\mathrm{H}_\lambda(a_1),\dots,\mathrm{H}_\lambda(a_n)\}\ \text{ for } a=(a_1,\dots,a_n)\in(\mathbb{Q}\lambda)^n.$$

For $Y\subseteq\mathbb{R}^n$ we introduce its finite subsets $Y_\lambda(T)$ and their cardinalities:

$$Y_\lambda(T)\ :=\ \{a\in Y\cap(\mathbb{Q}\lambda)^n:\ \mathrm{H}_\lambda(a)\leqslant T\},\qquad \mathrm{N}_\lambda(Y,T)\ :=\ \#Y_\lambda(T).$$

Theorem 8.5. *Let any definable $Y \subseteq \mathbb{R}^n$ and any ε be given. Then there is a constant $c = c(Y, d, \varepsilon) \in \mathbb{R}^{>}$ such that for all T and all $\lambda \in \mathbb{R}^d$,*

$$\mathrm{N}_\lambda(Y^{\mathrm{tr}}, T) \ \leqslant \ cT^\varepsilon.$$

Proof of Theorem 8.5. First a useful lemma about blocks:

Lemma 8.6. *If B is a block in $\mathbb{R}^n$ (of some dimension) and $p, q \in B$, then $\gamma(0) = p$ and $\gamma(1) = q$ for some continuous semialgebraic path $\gamma : [0,1] \to B$.*

Proof. Even better, let B be a connected open subset of a semialgebraic set $A \subseteq \mathbb{R}^n$, and let $p \in B$. We claim: there is for every $q \in B$ a continuous semialgebraic path $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma([0,1]) \subseteq B$. To see this, let $B(p)$ be the set of all $q \in B$ for which there is such a path. The sets $B(p)$ as p ranges over B form a partition of B, so it is enough to show that the $B(p)$ are open in B, which reduces to showing that $B(p)$ is a neighborhood of p in B. Now B is open in A, so we have a semialgebraic open subset U of A with $p \in U \subseteq B$. The connected component C of U with $p \in C$ is open in U and semialgebraic by Corollary A.3 and the remarks preceding it. These remarks also give $C \subseteq B(p)$. □

Corollary 8.7. *If B is a block in $\mathbb{R}^m$ (of some dimension), A is a semialgebraic subset of $\mathbb{R}^m$ with $B \subseteq A$, and $\phi : A \to \mathbb{R}^n$ is a continuous semialgebraic map such that $\phi(B)$ has more than one point, then $\phi(B) = \phi(B)^{\mathrm{alg}}$.*

Proof. Use that the ϕ-image of a path γ as in Lemma 8.6 is a connected semialgebraic subset of $\phi(B)$. □

The next result is basically a consequence of Theorem 8.4, as the proof will show.

Theorem 8.8. *Given ε, there are a natural number $N = N(X, d, \varepsilon) \geqslant 1$, a definable set $V_j \subseteq (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X, d, \varepsilon)$, such that for $j = 1, \dots, N$ and all $(s, \lambda, t) \in E \times \mathbb{R}^d \times F_j$:*

(i) *$V_j(s, \lambda, t) \subseteq X(s)$ and $V_j(s, \lambda, t)$ is connected;*
(ii) *if $\dim V_j(s, \lambda, t) \geqslant 1$, then $V_j(s, \lambda, t) \subseteq X(s)^{\mathrm{alg}}$,*

and such that for all T and $(s, \lambda) \in E \times \mathbb{R}^d$, the set $X(s)_\lambda(T)$ is covered by at most cT^ε sections $V_j(s, \lambda, t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.

This yields a family version of Theorem 8.5 as follows. Let $V_1, \dots, V_N$ and c be as in Theorem 8.8. Then for all $s \in E$ the definable set $V(s) \subseteq \mathbb{R}^n$ given by

$$V(s) \ := \ \bigcup \{V_j(s, \lambda, t) :\ 1 \leqslant j \leqslant N,\ (\lambda, t) \in \mathbb{R}^d \times F_j,\ \dim V_j(s, \lambda, t) \geqslant 1\}$$

is contained in $X(s)^{\mathrm{alg}}$ and $\mathrm{N}_\lambda(X(s) \setminus V(s), T) \leqslant cT^\varepsilon$ for all T.

Proof. Let $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^n \to \mathbb{R}^n$ be given by $\pi(\lambda, a_1, \dots, a_n) = (\lambda \cdot a_1, \dots, \lambda \cdot a_n)$, where $a_1, \dots, a_n \in \mathbb{R}^d$. Set

$$X^* \ := \ \{(s, \lambda, a_1, \dots, a_n) \in (E \times \mathbb{R}^d) \times (\mathbb{R}^d)^n :\ \big(s, \pi(\lambda, a_1, \dots, a_n)\big) \in X\},$$

viewed as a definable family of subsets of $(\mathbb{R}^d)^n$. Note that for $s \in E$ and $\lambda \in \mathbb{R}^d$,

$$(*) \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)\big) \ \subseteq \ X(s), \qquad \pi\big(\{\lambda\} \times X^*(s, \lambda)(\mathbb{Q}, T)\big) \ = \ X(s)_\lambda(T).$$

We apply Theorem 8.4 to X^* in the role of X. It gives $N = N(X^*, \varepsilon) \geqslant 1$, a block family $V_j^* \subseteq (E \times \mathbb{R}^d \times F_j) \times (\mathbb{R}^d)^n$ in $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j = 1, \dots, N$, and a constant $c = c(X^*, \varepsilon)$ such that:

(i)* $V_j^*(s,\lambda,t) \subseteq X^*(s,\lambda)$ for $j=1,\dots,N$ and (s,λ,t) in $E \times \mathbb{R}^d \times F_j$;

(ii)* for all T and all $(s,\lambda) \in E \times \mathbb{R}^d$, the set $X^*(s,\lambda)(\mathbb{Q},T)$ is covered by at most cT^ε sections $V_j^*(s,\lambda,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.

Now we set for $j=1,\dots,N$,

$$V_j \ :=\ \{\big(s,\lambda,t,\pi(\lambda,a)\big) \in (E \times \mathbb{R}^d \times F_j) \times \mathbb{R}^n :\ (s,\lambda,t,a) \in V_j^*\},$$

so $V_j(s,\lambda,t) = \pi\big(\{\lambda\} \times V_j^*(s,\lambda,t)\big)$ for $(s,\lambda,t) \in E \times \mathbb{R}^d \times F_j$. We now show that $V_1,\dots,V_N$ and $c(X,d,\varepsilon) := c(X^*,\varepsilon)$ have the desired properties. Clause (i) is satisfied using (i)* and $(*)$, and (ii) is satisfied in view of Corollary 8.7. The rest follows from (ii)* and $(*)$. □

Extending the Counting Theorem to Algebraic Points. Throughout this subsection we fix $d \geqslant 1$. Instead of rational points we now count algebraic points whose coordinates are of degree at most d over $\mathbb{Q}$. We define the corresponding height of an algebraic number $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant d$ by

$$\mathrm{H}_d^{\mathrm{poly}}(\alpha) \ :=\ \min\{\mathrm{H}(\xi) :\ \xi \in \mathbb{Q}^d,\ \alpha^d + \xi_1\alpha^{d-1} + \cdots + \xi_d = 0\} \in \mathbb{N}^{\geqslant 1}.$$

(For us this height is notationally more convenient than the height for real algebraic numbers used by Pila in [P2]. The two heights are related as follows, where we use an extra subscript P for the height in [P2]: for $\alpha \in \mathbb{R}$ with $[\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant d$,

$$\mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha) \ \leqslant\ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \ \leqslant\ \mathrm{H}_{\mathrm{P},d+1}^{\mathrm{poly}}(\alpha)^2.$$

Thus the results below for our height also hold for the other height.)

We extend the above height to all $\alpha \in \mathbb{R}$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \infty$ if $[\mathbb{Q}(\alpha):\mathbb{Q}] > d$, and to all points $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$ by $\mathrm{H}_d^{\mathrm{poly}}(\alpha) := \max\{\mathrm{H}_d^{\mathrm{poly}}(\alpha_1),\dots,\mathrm{H}_d^{\mathrm{poly}}(\alpha_n)\}$. For $Y \subseteq \mathbb{R}^n$ we introduce its finite subsets $Y_d(T)$ and their cardinalities:

$$Y_d(T) \ :=\ \{\alpha \in Y :\ \mathrm{H}_d^{\mathrm{poly}}(\alpha) \leqslant T\}, \qquad \mathrm{N}_d(Y,T) \ :=\ \#Y_d(T).$$

Theorem 8.9. *Let $Y \subseteq \mathbb{R}^n$ be definable, and let ε be given. Then there is a constant $c = c(Y,d,\varepsilon)$ such that for all T,*

$$N_d(Y^{\mathrm{tr}},T) \ \leqslant\ cT^\varepsilon.$$

We shall use the following easy consequence of semialgebraic cell decomposition:

Lemma 8.10. *Let $A_{n,d} \subseteq \mathbb{R}^{n\times d} \times \mathbb{R}^n$ be the semialgebraic set*

$$\{(\xi,\alpha) \in \mathbb{R}^{n\times d} \times \mathbb{R}^n :\ \alpha_i^d + \xi_{i1}\alpha_i^{d-1} + \cdots + \xi_{id} = 0 \text{ for } i=1,\dots,n\}.$$

Then we have a natural number $L = L(n,d) \geqslant 1$, a semialgebraic set $D_l \subseteq \mathbb{R}^{n\times d}$ with a semialgebraic continuous map $\phi_l : D_l \to \mathbb{R}^n$, for $l = 1,\dots,L$, such that $A_{n,d} = \bigcup_{l=1}^L \mathrm{graph}(\phi_l)$. It follows that for all $\alpha \in \mathbb{R}^n$ with $\mathrm{H}_d^{\mathrm{poly}}(\alpha) < \infty$ there is an $l \in \{1,\dots,L\}$ and a $\xi \in D_l$ such that $\phi_l(\xi) = \alpha$ and $\mathrm{H}(\xi) = \mathrm{H}_d^{\mathrm{poly}}(\alpha)$.

Towards Theorem 8.9 we first prove something stronger:

Theorem 8.11. *Let ε be given. Then there are $N = N(X,d,\varepsilon) \in \mathbb{N}^{\geqslant 1}$, a definable set $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ with definable $F_j \subseteq \mathbb{R}^{m_j}$, for $j=1,\dots,N$, and a constant $c = c(X,d,\varepsilon)$, such that for $j=1,\dots,N$ and all $(s,t) \in E \times F_j$:*

(i) *$V_j(s,t) \subseteq X(s)$ and $V_j(s,t)$ is connected;*

(ii) *if $\dim V_j(s,t) \geqslant 1$, then $V_j(s,t) \subseteq X(s)^{\mathrm{alg}}$,* *and such that for all T and $s \in E$, the set $X(s)_d(T)$ is covered by at most cT^{ε} sections $V_j(s,t)$, $(1 \leqslant j \leqslant N,\ t \in F_j)$.*

Proof. Let $\pi : \mathbb{R}^{n\times d} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be the obvious projection map. Take L and $\phi_1 : D_1 \to \mathbb{R}^n, \dots, \phi_L : D_L \to \mathbb{R}^n$ as in Lemma 8.10. Let $l \in \{1,\dots,L\}$. We set

$$\begin{aligned} X_l &:= \{(s,\xi,\alpha) \in E \times D_l \times \mathbb{R}^n :\ \alpha \in X(s),\ \phi_l(\xi) = \alpha\}, \\ Y_l &:= \{(s,\xi) \in E \times D_l :\ \xi \in \pi\big(X_l(s)\big)\} \ =\ \{(s,\xi) \in E \times D_l :\ \phi_l(\xi) \in X(s)\}, \end{aligned}$$

so for $s \in E$ we have $\phi_l\big(Y_l(s)\big) \subseteq X(s)$, and by Lemma 8.10, for all T,

$$X(s)_d(T) \ =\ \bigcup_{l=1}^{L} \phi_l\big(Y_l(s)(\mathbb{Q},T)\big).$$

We now apply Theorem 8.4 to Y_l in the role of X, and get $N_l = N_l(Y_l,\varepsilon) \in \mathbb{N}^{\geqslant 1}$, a block family $V_{l,i} \subseteq (E \times F_{l,i}) \times \mathbb{R}^{n\times d}$ in $\mathbb{R}^{n\times d}$ with definable $F_{l,i} \subseteq \mathbb{R}^{m_{l,i}}$, for $i = 1,\dots,N_l$, and a constant $c_l = c_l(Y_l,\varepsilon) \in \mathbb{R}^{>}$ such that:

(i) $V_{l,i}(s,t) \subseteq Y_l(s)$ for $i = 1,\dots,N_l$ and (s,t) in $E \times F_{l,i}$;

(ii) for all T and all $s \in E$, the set $Y_l(s)(\mathbb{Q},T)$ is covered by at most $c_l T^{\varepsilon}$ blocks $V_{l,i}(s,t)$, $(1 \leqslant i \leqslant N_l,\ t \in F_{l,i})$.

Set $N := N_1 + \cdots + N_L$, and for $1 \leqslant i \leqslant N_l$ and $j = N_1 + \cdots + N_{l-1} + i$, set $F_j := F_{l,i}$, and let $V_j \subseteq (E \times F_j) \times \mathbb{R}^n$ be the definable set given by

$$V_j(s,t) \ :=\ \phi_l\big(V_{l,i}(s,t)\big), \qquad (s \in E,\ t \in F_j).$$

It is easily verified using Lemma 8.7 that $V_1,\dots,V_N$ and $c(X,d,\varepsilon) := c_1 + \cdots + c_L$ have the properties stated in the Theorem. □

Just as with Theorem 8.8 this leads to a family version of Theorem 8.9 as follows. Let $V_1,\dots,V_N$ and c be as in Theorem 8.11. Take the definable set $V \subseteq E \times \mathbb{R}^n$ such that for all $s \in E$,

$$V(s) \ :=\ \bigcup\{V_j(s,t) :\ 1 \leqslant j \leqslant N,\ t \in F_j,\ \dim V_j(s,t) \geqslant 1\}.$$

Then for all $s \in E$ and all T we have

$$V(s) \ \subseteq\ X(s)^{\mathrm{alg}} \quad \text{and} \quad \mathrm{N}_d(X(s) \setminus V(s), T) \ \leqslant\ cT^{\varepsilon}.$$

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APPENDIX ON O-MINIMALITY

O-minimality as a subject started in [3] and [9]. Here we focus on material used in this paper. We give the key definitions in full detail, with examples, but state most results without proof. These proofs are in [5] as to general facts about o-minimal fields and the semialgebraic case, and in [2, 4, 6, 7, 10, 11] as to specific examples beyond the semialgebraic case. Notation is as in the introduction to this paper, in particular, $l, m, n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

We have divided this appendix into two parts. Part A is enough for sections 1–5, but sections 6 and 7 require some model-theoretic compactness, *alias* saturation, which is fully exposed in part B.

A. O-minimal fields

Structures. Let M be a nonempty set. We consider the finite cartesian powers

$$M^n \ := \ \{a = (a_1, \dots, a_n) : \ a_1, \dots, a_n \in M\},$$

identifying in the usual way M^1 with M and M^{m+n} with $M^m \times M^n$. A *structure on* M is a sequence $\mathcal{S}{=}(\mathcal{S}_n)$ such that for all n,

(1) $\mathcal{S}_n$ is a boolean algebra of subsets of M^n, that is, all $X \in \mathcal{S}_n$ are subsets of M^n, $M^n \in \mathcal{S}_n$, and for all $X, Y \in \mathcal{S}_n$ also $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{S}_n$.
(2) For $n \geqslant 2$ and $1 \leqslant i < j \leqslant n$ the diagonal $\{a \in M^n : a_i = a_j\} \in \mathcal{S}_n$.
(3) If $X \in \mathcal{S}_n$, then $M \times X \in \mathcal{S}_{n+1}$ and $X \times M \in \mathcal{S}_{n+1}$.
(4) If $X \in \mathcal{S}_{n+1}$, then $\pi(X) \in \mathcal{S}_n$, where $\pi : M^{n+1} \to M^n$ is the projection map given by $\pi(a_1, \dots, a_n, a_{n+1}) = (a_1, \dots, a_n)$.

Let $\mathcal{S}$ be a structure on M. The definition of "structure" lacks symmetry, but in fact, if $X \in \mathcal{S}_n$ and σ is a permutation of $\{1, \dots, n\}$, then

$$\{\big(a_{\sigma(1)}, \dots, a_{\sigma(n)}\big) : \ a = (a_1, \dots, a_n) \in X\} \in \mathcal{S}_n.$$

Given a map $f : X \to M^n$ with $X \subseteq M^m$, we say that f *belongs to* $\mathcal{S}$ (or $\mathcal{S}$ *contains* f) if its graph, as a subset of M^{m+n}, belongs to $\mathcal{S}_{m+n}$; in that case $X \in \mathcal{S}_m$, $f(X) \in \mathcal{S}_n$, $f^{-1}(Y) \in \mathcal{S}_m$ for every $Y \in \mathcal{S}_n$, and the restriction $f|_{X_0} : X_0 \to R^n$ belongs to $\mathcal{S}$ for every $X_0 \subseteq X$ in $\mathcal{S}_m$. If $f : X \to M^n$ and $g : Y \to M^l$ belong to $\mathcal{S}$, where $X \subseteq M^m$ and $Y \subseteq M^n$, then the composition $g \circ f : X \cap f^{-1}(Y) \to M^l$ belongs to $\mathcal{S}$. The class of all structures on M is partially ordered by $\subseteq$:

$$\mathcal{S} \subseteq \mathcal{S}' \ :\Longleftrightarrow \ \mathcal{S}_n \subseteq \mathcal{S}'_n \text{ for all } n.$$

Any collection $\mathcal{C}$ of sets $X \subseteq M^n$ for various n gives rise to the *least* structure $\mathcal{S}$ on M that contains every $X \in \mathcal{C}$, where "least" is with respect to $\subseteq$.

Ordered fields. Let R be an ordered field: a field with a (strict) total order $<$ on its underlying set such that for all $a, b, c \in R$ we have

$$a < b \Rightarrow a + c < b + c, \qquad a < b, \ 0 < c \Rightarrow ac < bc.$$

The case to keep in mind is the field $\mathbb{R}$ of real numbers with its usual ordering, but in sections 4–8 we work in bigger ambient ordered fields, since results in that setting have consequences for $\mathbb{R}$ that are less easy to obtain otherwise. (This is where model theory comes into play.) The ordered field $\mathbb{Q}$ of rational numbers embeds uniquely into R as an ordered field. We use also the signs $\leqslant, >, \geqslant$ with the usual meaning derived from $<$, and set $R^{>} := \{a \in R : a > 0\}$, $R^{\geqslant} := \{a \in R : a \geqslant 0\}$. For $a \in R$ we set $|a| := a$ if $a \geqslant 0$ and $|a| := -a$ if $a \leqslant 0$. For $a = (a_1, \ldots, a_n) \in R^n$ we set $|a| := \max(|a_1|, \ldots, |a_n|) \in R^{\geqslant}$, which by convention equals 0 if $n = 0$.

An *interval* in R is a set $(a, b) := \{x \in R : a < x < b\}$, where $a, b \in R_\infty := R \cup \{-\infty, \infty\}$, $a < b$, where we extend $<$ to a total ordering on R_∞ by $-\infty < x < \infty$ for all $x \in R$. For $a \leqslant b$ in R_∞ we also set $[a, b] := \{x \in R_\infty : a \leqslant x \leqslant b\}$, but we do not call this an interval. We endow R with the order topology on its underlying set: it has the collection of intervals as a basis, and is a hausdorff topology. We also equip R^n with the corresponding product topology.

We call R *real closed* if $R^{>} = \{b^2 : 0 \neq b \in R\}$ and every polynomial $p(x) \in R[x]$ of odd degree has a zero in R. (This is equivalent to the field $R[\mathrm{i}]$ with $i^2 = -1$ being algebraically closed.) In particular, the ordered field $\mathbb{R}$ of real numbers is real closed, and in some precise sense, all real closed fields have the same elementary properties as $\mathbb{R}$ (Tarski); we do not explicitly use that fact. Here and below $\mathbb{R}$ denotes the *ordered field* of real numbers, not just the set of real numbers.

O-Minimal Structures. Let R again be an ordered field. A *structure on* R is a structure $\mathcal{S}$ on its underlying set such that

(5) $\{(a, b) \in R^2 : a < b\} \in \mathcal{S}_2$ and the graphs of $+, \cdot : R^2 \to R$ lie in $\mathcal{S}_3$.

Let $\mathcal{S}$ be a structure on R with $\{a\} \in \mathcal{S}_1$ for all $a \in R$. Then every interval is in $\mathcal{S}_1$, for every polynomial $p \in R[x_1, \ldots, x_n]$ the corresponding function $a \mapsto p(a) : R^n \to R$ belongs to $\mathcal{S}$, and so $\mathcal{S}_n$ contains the sets

$$\{a \in R^n : p(a) = 0\} \text{ and } \{a \in R^n : p(a) > 0\}.$$

For real closed R, a *semialgebraic subset of* R^n is a finite union of sets

$$\{a \in R^n : p(a) = 0,\ q_1(a) > 0, \ldots, q_m(a) > 0\}, \quad (p, q_1, \ldots, q_m \in R[x_1, \ldots, x_n]),$$

and setting $\mathcal{S}_n := \{\text{semialgebraic subsets of } R^n\}$ gives by the Tarski-Seidenberg theorem a structure $\mathcal{S} = (\mathcal{S}_n)$ on the ordered field R. This is the least structure on R containing $\{a\}$ for all $a \in R$. In this case $\mathcal{S}_1$ contains exactly the finite unions of the sets $\{a\}$ with $a \in R$ and intervals. This fact about $\mathcal{S}_1$ is a surprisingly strong minimality property of $\mathcal{S}$ which we now axiomatize:

An *o-minimal structure on* R is a structure on the ordered field R such that

(6) $\{a\} \in \mathcal{S}_1$ for every $a \in R$ and every element of $\mathcal{S}_1$ is a finite union of one-element subsets of R and intervals.

One can show that R must be real closed if there is an o-minimal structure on it. The theory of o-minimal structures is a wide ranging generalization of the older subject of semialgebraic sets, and much of the tame properties of semialgebraic sets go through for the sets belonging to an o-minimal structure, as we shall see. The significance of o-minimality for applications is largely due to the fact that there are interesting o-minimal structures on $\mathbb{R}$ beyond its structure of semialgebraic sets. The important examples below are by way of illustration; the general facts about o-minimal structures that we focus on in this part of the appendix do not depend on the nontrivial theorems that establish the o-minimality of these examples.

Terminology: an *o-minimal field* is an ordered field equipped with an o-minimal structure on it (and this ordered field is then real closed). We let $\mathcal{S}_{\text{alg}}$ be the o-minimal structure on $\mathbb{R}$ consisting of the semialgebraic subsets of $\mathbb{R}^n$, for all n. The first examples of o-minimal fields beyond the semialgebraic case are:

(i) $\mathbb{R}_{\mathrm{an}}$: this is $\mathbb{R}$ equipped with the smallest structure $\mathcal{S}_{\mathrm{an}}$ on it that contains every $f : [-1,1]^n \to \mathbb{R}$ that extends to a real analytic function $U \to \mathbb{R}$ on some open neighborhood $U \subseteq \mathbb{R}^n$ of $[-1,1]^n$, for $n = 0,1,2,\dots$. A set $X \subseteq \mathbb{R}^n$ belongs to $\mathcal{S}_{\mathrm{an}}$ iff X is subanalytic in the larger (compact) real analytic manifold $\mathbb{P}(\mathbb{R})^n$, where $\mathbb{P}(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is the real projective line. The study of $\mathbb{R}_{\mathrm{an}}$ is essentially the theory of subanalytic sets due to Hironaka and Gabrielov: see [2, 4].

(ii) $\mathbb{R}_{\exp}$: this is $\mathbb{R}$ with the smallest structure $\mathcal{S}_{\exp}$ on it containing $\{r\}$ for all $r \in \mathbb{R}$, and the function $\exp : \mathbb{R} \to \mathbb{R}$, $\exp(r) := \mathrm{e}^r$. A set $X \subseteq \mathbb{R}^m$ belongs to $\mathcal{S}_{\exp}$ iff $X = \pi\big(\{a \in \mathbb{R}^n : \ P(a, \mathrm{e}^a) = 0\}\big)$ for some $n \geqslant m$ and some polynomial $P \in \mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n]$, where $\mathrm{e}^a := (\mathrm{e}^{a_1},\dots,\mathrm{e}^{a_n})$ and $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is given by $\pi(a) = (a_1,\dots,a_m)$ for $a = (a_1,\dots,a_n) \in \mathbb{R}^n$. This characterization of $\mathcal{S}_{\exp}$ is part of Wilkie's theorem in [11].

(iii) For applications in arithmetic algebraic geometry it is important that we can amalgamate (i) and (ii) into an o-minimal field $\mathbb{R}_{\mathrm{an,exp}}$: this is $\mathbb{R}$ with the smallest structure $\mathcal{S}_{\mathrm{an,exp}}$ on it such that $\mathcal{S}_{\mathrm{an,exp}} \supseteq \mathcal{S}_{\mathrm{an}}, \mathcal{S}_{\exp}$. A characterization of $\mathcal{S}_{\mathrm{an,exp}}$ in the style of (ii) is in [6], and a sharper one in [7] where also the description of $\mathcal{S}_{\mathrm{an}}$ in (i) is improved.

In general, amalgamation as in Example (iii) does not preserve o-minimality: [10] describes two o-minimal structures $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathbb{R}$ for which the smallest structure $\mathcal{S}$ on $\mathbb{R}$ with $\mathcal{S} \supseteq \mathcal{S}_1, \mathcal{S}_2$ is not o-minimal.

As to the appearance of exponentiation in the examples above, [8] proves a striking dichotomy: for any o-minimal structure $\mathcal{S}$ on $\mathbb{R}$, either exp belongs to $\mathcal{S}$, or every function $\mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{S}$ is polynomially bounded, as $t \to \infty$. It is not known if there exists an o-minimal structures $\mathcal{S}$ on $\mathbb{R}$ with a function $\mathbb{R} \to \mathbb{R}$ belonging to it that grows faster, as $t \to \infty$, than any finite iterate of the exponential function.

One way that o-minimality can fail (very badly) for a structure $\mathcal{S}$ on $\mathbb{R}$ with $\{a\} \in \mathcal{S}_1$ for all $a \in \mathbb{R}$ is that $\mathbb{Z} \in \mathcal{S}_1$: one can show that then all closed subsets of all $\mathbb{R}^n$ belong to $\mathcal{S}$, and even the Lebesgue-measurability of certain sets in $\mathcal{S}$ cannot be settled without unorthodox set-theoretic axioms. In particular, the sine function on $\mathbb{R}$ cannot belong to any o-minimal structure on $\mathbb{R}$, although its restriction to any bounded interval belongs to the o-minimal structure $\mathcal{S}_{\mathrm{an}}$ on $\mathbb{R}$.

Definable Sets. In the rest of part A of the appendix we fix an o-minimal field R. Its underlying real closed ordered field is also denoted by R. For a set $X \subseteq R^m$ we call X *definable* if X belongs to the given o-minimal structure of R, and likewise for maps $X \to R^n$. (This use of the term "definable" has its origin in logic, for which see part B of this appendix.) In case the given o-minimal structure on R consists just of the semialgebraic sets (in the sense of the real closed field R), we write *semialgebraic* in place of *definable*.

Topological notions like openness and continuity are with respect to the order topology on R and the corresponding product topology on each R^n. If $X \subseteq R^n$ is definable, then so are its closure $\mathrm{cl}(X)$ and its interior $\mathrm{int}(X)$ in $\mathbb{R}^n$. The definable homeomorphism $t \mapsto \frac{t}{1+|t|} \ : \ R \to (-1,1)$ extends to an order preserving bijection $R_\infty \to [-1,1]$ sending $-\infty$ to -1 and ∞ to 1, and we equip R_∞ with the (hausdorff) topology on it making this bijection into a homeomorphism.

Till further notice the results below are from [5, Chapter 3], where the o-minimal structures considered are more general, with just an underlying nonempty totally ordered set without least or greatest element and such that for any two distinct elements $a < b$ there is an x with $a < x < b$, no field operations being included.

Here is the key fact about univariate definable functions:

Theorem A.1 (Monotonicity Theorem)**.** *Let $I = (a, b)$ be an interval and let $f : (a, b) \to R$ be definable. Then f has the following properties:*

- (i) *there are points $a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ such that on each subinterval (a_j, a_{j+1}) with $0 \leqslant j \leqslant n$ the function f is continuous, and either strictly decreasing, or constant, or strictly increasing.*
- (ii) *if f is continuous and $f(p) < c < f(q)$ with $p < q$ in I, then $c = f(x)$ for some $x \in (p, q)$. (Intermediate Value Property.)*
- (iii) $\lim_{t\downarrow a} f(t)$ *and* $\lim_{t\uparrow b} f(t)$ *exists in R_∞.*

Of course, the intermediate value property (ii) is automatic when the underlying ordered field is $\mathbb{R}$ and then requires no definability assumption. In the o-minimal setting, and certainly outside the familiar real environment, we confine attention to definable objects. For example, the correct analogue of "connected" is as follows: a definable set $X \subseteq R^m$ is said to be *definably connected* if there are no disjoint nonempty definable open subsets X_0, X_1 of X with $X = X_0 \cup X_1$. For such X and any definable continuous map $f : X \to R^n$, the image $f(X) \subseteq R^n$ is also definably connected. Intervals are definably connected.

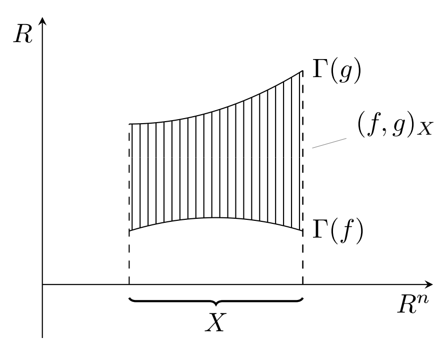
Cells. Towards partitioning an arbitrary definable set $X \subseteq R^n$ into finitely many nice pieces we introduce *cells*. These are definably connected sets of a form that makes them suited to proofs by induction (on n, for cells in R^n). First some notation. Let $X \subseteq R^n$ be definable. Set

$$C(X) := \{f : X \to R : f \text{ is definable and continuous}\},$$
$$C_\infty(X) := C(X) \cup \{-\infty, \infty\},$$

where $-\infty$ and ∞ are viewed as constant functions on X. Let $f, g \in C_\infty(X)$, and suppose $f < g$, that is, $f(x) < g(x)$ for all $x \in X$. Then we set

$$(f, g) = (f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\},$$

so $(f, g) \subseteq R^{n+1}$ is definable; see next picture.



Let $n \geqslant 1$ and $(i_1, \ldots, i_n)$ a sequence of zeros and ones. An $(i_1, \ldots, i_n)$-cell is a definable subset of R^n obtained via the following recursion:

(i) Case $n = 1$: a (0)-cell is a one-element subset of R, a (1)-cell is an interval;
(ii) An $(i_1, \dots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$ on an $(i_1, \dots, i_n)$-cell X; an $(i_1, \dots, i_n, 1)$-cell is a set $(f, g)_X$ with $f, g \in C_\infty(X)$, $f < g$, and X an $(i_1, \dots, i_n)$-cell.

A *cell in* R^n is an $\boldsymbol{i}$-cell, for some (necessarily unique) $\boldsymbol{i} = (i_1, \dots, i_n) \in \{0, 1\}^n$. A cell in R^n is definably connected, and locally closed (open in its closure in R^n). A cell in R^n is open in R^n (and called an *open cell*) iff it is a $(1, \dots, 1)$-cell.

Important in sections 2 and 8 is that every cell is homeomorphic under a coordinate projection to an open cell. In detail, let $C \subseteq R^n$ be an $\boldsymbol{i}$-cell, $\boldsymbol{i} = (i_1, \dots, i_n)$. Let $\lambda(1) < \dots < \lambda(k)$ be the indices $\lambda \in \{1, \dots, n\}$ with $i_\lambda = 1$, and consider the (definable) coordinate projection $p_{\boldsymbol{i}} : R^n \to R^k$ given by

$$p_{\boldsymbol{i}}(x_1, \dots, x_n) = (x_{\lambda(1)}, \dots, x_{\lambda(k)}).$$

Then $p_{\boldsymbol{i}}$ maps C homeomorphically onto an open cell in R^k. We denote this open cell $p_{\boldsymbol{i}}(C)$ also by $p(C)$ and the homeomorphism $p_{\boldsymbol{i}}|_C : C \to p(C)$ by p_C.

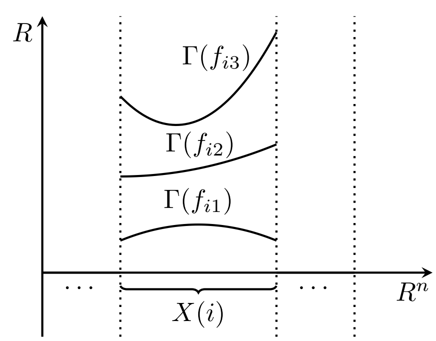
Cell Decomposition. Let $n \geqslant 1$. A *decomposition of* R^n is a partition of R^n into finitely many cells, obtained by the following recursion:

(i) case $n = 1$: points $a_1 < \dots < a_m$ in R determine a decomposition of $R = R^1$ consisting of $(-\infty, a_1)$, $\{a_1\}$, $(a_1, a_2), \dots, (a_{m-1}, a_m)$, $\{a_m\}$, (a_m, ∞).
(ii) a decomposition $\mathcal{D}$ of R^{n+1} is a finite partition of R^{n+1} into cells such that $\pi(\mathcal{D}) := \{\pi(C) : C \in \mathcal{D}\}$ is a decomposition of R^n, where $\pi : R^{n+1} \to R^n$ is the projection map given by $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$.

With $X(1), \dots, X(k)$ the distinct cells of a decomposition $\mathcal{D}$ of R^n, let functions $f_{i1} < \dots < f_{im_i}$ in $C(X_i)$ be given for $i = 1, \dots, k$. Then

$$\mathcal{D}_i = \{(-\infty, f_{i1}), \Gamma(f_{i1}), (f_{i1}, f_{i2}), \dots, \Gamma(f_{im_i}), (f_{im_i}, \infty)\}$$

is a partition of $X(i) \times R$, and $\mathcal{D}^* = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_k$ is a decomposition of R^{n+1} with $\mathcal{D} = \pi(\mathcal{D}^*)$. See the figure below. Every decomposition of R^{n+1} is obtained in this manner from a decomposition of R^n.



In these definitions of *cell* and *decomposition* we assumed $n \geqslant 1$, but it is convenient to also consider the one-point set R^0 as the unique cell in R^0, namely as an $\boldsymbol{i}$-cell where $\boldsymbol{i} \in \{0, 1\}^0$ is the empty tuple of zeros and ones, and $\{R^0\}$ as the unique decomposition of R^0. In this way clause (i) in these definitions appears as the case $n = 0$ of the corresponding clause (ii). So below we allow $n = 0$.

A decomposition $\mathcal{D}$ of R^n is said to *partition* a set $X \subseteq R^n$ if each cell in $\mathcal{D}$ is either contained in X or disjoint from X (so X is a union of cells in $\mathcal{D}$). We can now state the fundamental Cell Decomposition Theorem:

Theorem A.2. *For any definable $X_1, \dots, X_m \subseteq R^n$ some decomposition of R^n partitions $X_1, \dots, X_m$. If $X \subseteq R^n$ and $f : X \to R$ are definable, then some decomposition $\mathcal{D}$ of R^n partitions X with continuous $f|_C$ for all cells $C \subseteq X$ in $\mathcal{D}$.*

Some consequences: if the definable set $X \subseteq R^n$ is definably connected, then it is "definably path connected": for any points $p, q \in X$ there is a definable continuous $\gamma : [0,1] \to X$ with $\gamma(0) = p, \gamma(1) = q$. If the underlying ordered field of R is $\mathbb{R}$, then for definable $X \subseteq R^n$, *definably connected* agrees with *connected*.

A *definably connected component* of a definable set $X \subseteq R^n$ is a definably connected definable nonempty subset of X that is maximal with respect to inclusion. (So if $X = \emptyset$, it has no definably connected components.)

Corollary A.3. *For definable $X \subseteq R^n$, the definably connected components of X are all open and closed in X, and form a finite partition of X.*

Definable families. Let $E \subseteq R^m$ and $X \subseteq E \times R^n \subseteq R^{m+n}$ be definable. For $a \in E$ we set

$$X(a) \ := \ \{x \in R^n : \ (a, x) \in X\}.$$

We view X as describing the family $\big(X(a)\big)_{a \in E}$ of definable subsets of R^n. We call this a *definable family*, and the sections $X(a)$ are the members of the family.

Example. The hypersurfaces in R^2 of degree at most 2 are the members of a semialgebraic family: such a hypersurface is the set of solutions in R^2 of an equation

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \ = \ 0 \quad \text{ with } (a_1, a_2, a_2, a_4, a_5, a_6) \in \mathbb{R}^6 \setminus \{0\},$$

so here $E = R^6 \setminus \{0\}$ and X consists of the points $(a_1, a_2, a_2, a_4, a_5, a_6, x, y) \in E \times R^2$ satisfying the above equation. By the same token, for any n and d the hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant d$ are the members of a semialgebraic family.

Let $\pi : R^{m+n} \to R^m$ be given by $\pi(x_1, \dots, x_{m+n}) = (x_1, \dots, x_m)$.

Proposition A.4. *Suppose $\mathcal{D}$ is a decomposition of R^{m+n} partitioning X. Then for each $a \in E$,*

$$\mathcal{D}(a) := \{C(a) : C \in \mathcal{D}, a \in \pi(C)\}$$

is a decomposition of R^n partitioning $X(a)$. This gives in particular a finite bound on the number of definably connected components of $X(a)$ independent of $a \in E$.

Dimension. This subsection is taken from [5, Chapter 4, section 1]. It is natural to assign to an $(i_1, \dots, i_n)$-cell C the dimension

$$\dim C \ := \ i_1 + \cdots + i_n \in \{0, \dots, n\},$$

since $p_{(i_1,\dots,i_n)} : R^n \to R^{i_1+\cdots+i_n}$ is definable and maps C homeomorphically onto an open subset of $R^{i_1+\cdots+i_n}$. Such C does not contain any $(j_1, \dots, j_n)$-cell with $j_1 + \cdots + j_n > \dim(C)$. This fact allows us to extend the above dimension to arbitrary nonempty definable $X \subseteq R^n$ by

$$\dim X \ := \ \max\{\dim C : \ C \subseteq X \text{ is a cell}\} \in \mathbb{N}.$$

We also set $\dim \emptyset := -\infty$. Here are some basic facts on dimension:

Proposition A.5. *Let $X \subseteq R^m$ be definable. Then:*

(i) $\dim X = 0 \iff X$ *is finite and nonempty;*
(ii) $\dim X = m \iff X$ *has nonempty interior in* R^m*;*
(iii) *if* $Y \subseteq R^m$ *is definable, then* $\dim X \cup Y = \max(\dim X, \dim Y)$*;*
(iv) *if* $Y \subseteq R^n$ *is definable, then* $\dim X \times Y = \dim X + \dim Y$*;*
(v) *if* $f : X \to R^n$ *is definable, then* $\dim X \geqslant \dim f(X)$*;*
(vi) *if* $f : X \to R^n$ *is definable and injective, then* $\dim X = \dim f(X)$*;*
(vii) *if* $X \neq \emptyset$*, then* $\dim(\mathrm{cl}(X) \setminus X) < \dim X$*.*

In (v), (vi) we do not assume f is continuous. Here is a stronger version of (v):

Proposition A.6. *Let* $f : X \to R^n$ *be definable,* $X \subseteq R^m$*. For* $d \leqslant m$*, set*

$$Y(d) \ := \ \{y \in R^n : \dim f^{-1}(y) = d\}.$$

Then $Y(d)$ *is definable and* $\dim X = \max_{d \leqslant m} d + \dim Y(d)$*.*

We also have a *local* dimension: Let $X \subseteq R^m$ be definable and $a \in R^m$. Then there is a definable neighborhood V of a in R^m such that $\dim(X \cap U) = \dim(X \cap V)$ for all definable neighborhoods $U \subseteq V$ of a in R^m; thus $\dim(X \cap V)$ is independent of the choice of such V, and we set $\dim_a X := \dim(X \cap V)$ for such V.

Definable Compactness. This subsection and the next are from [5, Chapter 6, section 1]. The ordinary notion of compactness from point set topology is useless in our setting, but we do have a good substitute. Call a set $X \subseteq R^m$ *bounded* if $X \subseteq [-r, r]^m$ for some $r \in R^{>}$.

Proposition A.7. *If* $f : X \to R^n$ *is a continuous definable map on a closed and bounded (definable) set* $X \subseteq R^m$*, then* $f(X) \subseteq R^n$ *is also closed and bounded.*

This has the expected consequences:

Corollary A.8. *If* $f : X \to R$ *is a continuous definable function on a nonempty closed bounded set* $X \subseteq R^m$*, then* f *has a maximum and a minimum value on* X*.*

Corollary A.9. *If* $f : X \to R^n$ *is an injective continuous definable map on a closed and bounded set* $X \subseteq R^m$*, then* $f : X \to f(X)$ *is a homeomorphism.*

Definable Selection. For any interval (a, b) we can "definably" select a point in it: $(a + b)/2$ if $a, b \in R$; $b - 1$ if $a = -\infty$ and $b \in R$; $a + 1$ if $a \in R$ and $b = \infty$; 0 if $a = -\infty$ and $b = \infty$. This can be exploited to give two very useful selection principles, the second a consequence of the first:

Proposition A.10. *Any definable equivalence relation on a definable set* $X \subseteq R^n$ *has a definable set of representatives, that is, a definable subset of* X *that has exactly one point in common with each equivalence class. Any definable map* $f : X \to R^n$*,* $X \subseteq R^m$*, has a definable right-inverse* $g : f(X) \to X$*, that is,* $f \circ g = \mathrm{id}_{f(X)}$*.*

In these last two subsections we got to use the underlying additive group of R, but not yet its multiplication. Accordingly this material goes through in the more general o-minimal setting of [5, Chapter 6] (not needed for our purpose). We now turn to a topic where multiplication does come into play.

The rest of part A is from [5, Chapter 7].

Differentiability. In this subsection we don't need o-minimality or definability, and R can be any ordered field. The elementary facts stated here have the same proofs as for $R = \mathbb{R}$. For $a, b \in R^n$ we set $a \cdot b := a_1 b_1 + \cdots + a_n b_n \in R$ (dot product). Let $I \subseteq R$ be open. A map $f : I \to R^n$ is said to be differentiable at a point $a \in I$ with derivative $b \in R^n$ if

$$\lim_{t \to 0} \frac{1}{t}\big(f(a+t) - f(a)\big) \ = \ b.$$

In that case f is continuous at a and we set $f'(a) := b$. If $f, g : I \to R^n$ are differentiable at a, then so are $f + g : I \to R^n$ and $f \cdot g : I \to R = R^1$, with

$$(f+g)'(a) \ = \ f'(a) + g'(a), \quad (f \cdot g)'(a) \ = \ f'(a) \cdot g(a) + f(a) \cdot g'(a),$$

and if in addition $n = 1$, g is continuous, and $g(a) \neq 0$, then $f/g : I \setminus g^{-1}(0) \to R$ is differentiable at a with $(f/g)'(a) = \big(f'(a)g(a) - f(a)g'(a)\big)/g(a)^2$. Constant maps $I \to R^n$ are differentiable at every $a \in I$ with derivative $0 \in R^n$, and the inclusion map $I \to R$ is differentiable at every $a \in I$ with derivative $1 \in R$.

Chain Rule: if $f : I \to R$ is continuous, differentiable at $a \in I$, and $f(a) \in J$ with open $J \subseteq R$, and $g : J \to R$ is differentiable at $f(a)$, then $g \circ f : I \cap f^{-1}(J) \to R$ is differentiable at a with $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Next we consider directional derivatives. We consider a map $f : U \to R^n$ with open $U \subseteq R^m$. For a point $a \in U$ and a vector $v \in R^m$ we say that f *is differentiable at a in the v-direction* if the R^n-valued map $t \mapsto f(a + tv)$ (defined on an open neighborhood of $0 \in R$) is differentiable at 0, that is, $\lim_{t \to 0} \frac{1}{t}\big(f(a+tv) - f(a)\big)$ exists in R^n, in which case we set

$$d_a f(v) \ := \ \lim_{t \to 0} \frac{1}{t}\big(f(a+tv) - f(a)\big) \in R^n.$$

For the standard basis vectors $e_1, \ldots, e_m$ of the R-linear space R^m we also write $\frac{\partial f}{\partial x_i}(a)$ for $d_a f(e_i)$.

Let $a \in U$ and let $T : R^m \to R^n$ be an R-linear map. We call f *differentiable at a with differential T* if for every $\varepsilon \in R^{>}$ we have, for all sufficiently small $v \in R^m$,

$$|f(a+v) - f(a) - T(v)| \ \leqslant \ \varepsilon |v|.$$

Then f is continuous at a and T is uniquely determined by f, a, so we can set $d_a f := T$, a notation consistent with that for directional derivatives: for each vector $v \in R^m$ the map f is differentiable at a in the v-direction with $d_a f(v) = T(v)$ where $d_a f(v)$ denotes the directional derivative defined earlier. For $m = 1$ this notion of differentiability at a agrees with the one defined earlier, with $d_a f(1) = f'(a)$.

The map $f = (f_1, \ldots, f_n) : U \to R^n$ is differentiable at a iff $f_1, \ldots, f_n : U \to R$ are differentiable at a. In that case all partials $\big(\partial f_i / \partial x_j\big)(a)$ exist and the $n \times m$ matrix $\big(\partial f_i / \partial x_j\big)(a)$ is the matrix of $d_a f$ with respect to the standard basis vectors of R^m and R^n. Each R-linear map $R^m \to R^n$ is differentiable at each point of R^m with itself as differential. If the maps $f, g : U \to R^n$ are differentiable at $a \in U$, then $f + g$ and cf for $c \in R$ are differentiable at a with

$$d_a(f+g) \ = \ d_a f + d_a g, \qquad d_a cf \ = \ c \cdot d_a f.$$

Chain Rule: Suppose $U \subseteq R^m$, $V \subseteq R^n$ are open, $a \in U$, $f : U \to R^n$ is continuous, f is differentiable at the point $a \in U$, $f(a) \in V$, and $g : V \to R^l$ is differentiable at

$f(a)$. Then $g \circ f : U \cap f^{-1}(V) \to R^l$ is differentiable at a, and

$$d_a(g \circ f) \;=\; \left(d_{f(a)}g\right) \circ d_a f.$$

Preserving Definability and the Mean Value Theorem. We now revert to the setting where R is an o-minimal field and our sets and maps are definable. So let $U \subseteq R^m$ be open and definable, and let $f : U \to R^n$ be definable. Then the set D_f of points (a,v) in $U \times R^m \subseteq R^{2m}$ such that f is differentiable at a in the v-direction is definable, and so is the map $(a,v) \mapsto d_a f(v) : D_f \to R^n$.

Lemma A.11. *Let $a < b$ in R, and suppose $f : [a,b] \to R$ is definable and continuous, and differentiable at each point of (a,b). Then there is $c \in (a,b)$ with*

$$f(b) - f(a) = f'(c) \cdot (b-a).$$

This "mean value" lemma opens the door to *continuous differentiability.* Let

$$f = (f_1, \ldots, f_n) : U \to R^n$$

be a definable map on a (definable) open set $U \subseteq R^m$. We say that f is *of class C^1* (or just C^1) if f is differentiable at every point $a \in U$ in the directions $e_1, \ldots, e_m$, and the resulting (definable) functions $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} : U \to R^n$ are continuous. In the next lemma we take matrices with respect to the standard bases of R^m and R^n.

Lemma A.12. *If f is C^1, then f is differentiable at each point of U and*

$$a \mapsto \text{matrix of } d_a f \;:\; U \to R^{n \times m}$$

is continuous. Conversely, if f is differentiable at each point of U and the above matrix-valued map is continuous, then f is a C^1-map.

For an R-linear map $T : R^m \to R^n$ we put $|T| := \max\{|Ta| : \ |a| \leqslant 1,\ a \in R^m\}$. (The maximum exists in R since T is definable and continuous.) Thus $|Ta| \leqslant |T| \cdot |a|$ for all $a \in R^m$. Now we can state an extended mean value result:

Lemma A.13. *Suppose $f : U \to R^n$ is of class C^1, and $a, b \in U$ are such that the line segment $[a,b] := \{(1-t)a + tb : \ 0 \leqslant t \leqslant 1\}$ is contained in U. Then*

$$|f(b) - f(a)| \leqslant |b-a| \cdot \max_{y \in [a,b]} |d_y f|.$$

Smooth Cell Decomposition. It is convenient to extend the notions of C^1-map and C^1-cell. We say that a definable map $f : X \to R^n$, $X \subseteq R^m$, is C^1 if there are a definable open $U \subseteq R^m$ such that $X \subseteq U$, and a definable C^1-map $F : U \to R^n$ such that $f = F|_X$. We define *C^1-cells* as in the recursive definition for *cells*, except that we require the (definable) functions f and g there, when R-valued, to be C^1 instead of just being continuous.

Every inclusion map $X \to R^m$ for definable $X \subseteq R^m$ is C^1. If the definable map $f : X \to R^n$ with $X \subseteq R^m$ is C^1 and the definable map $g : Y \to R^l$ with $Y \subseteq R^n$ is C^1, then $g \circ f : f^{-1}(Y) \to R^l$ is C^1. For definable $f = (f_1, \ldots, f_n) : X \to R^n$ with $X \subseteq R^m$, f is C^1 iff $f_1, \ldots, f_n$ are C^1.

The following is a C^1-version of cell decomposition.

Theorem A.14. *For any definable $X_1, \ldots, X_m \subseteq R^n$ there is a decomposition of R^n into C^1-cells partitioning $X_1, \ldots, X_m$. If $X \subseteq R^n$ and $f : X \to R$ are definable, then there is a decomposition of R^n into C^1-cells which partitions X such that $f|_C$ is C^1 for each cell $C \subseteq X$ of the decomposition.*

C^k-maps. Let $k \geqslant 1$ below. Let $f = (f_1, \dots, f_n) : U \to R^n$ be definable, with (definable) open $U \subseteq R^m$. By recursion on k we specify what it means for f to be a C^k-map. For $k = 1$ this has been defined earlier, and for $k > 1$ we say that f is C^k if f is C^1 and every partial $\partial f_i / \partial x_j : U \to R$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$ is C^{k-1}.

We also extend being C^k to definable maps $f : X \to R^n$ with $X \subseteq R^m$ not necessarily open, just as we did for $k = 1$, and likewise we define the notion of a C^k-cell just like we did for $k = 1$. The remarks we made above about this extended notion of definable C^1-map go through when "C^1" is replaced by "C^k" everywhere. The C^1-cell decomposition theorem goes through with "C^1" replaced by "C^k" everywhere.

Semialgebraic Functions. We finish part A of this appendix with some facts on real semialgebraic functions, and use this to determine the algebraic part of the set

$$X \ = \ \{(a,b,c) \in \mathbb{R}^3 :\ 1 < a, b < 2,\ c = a^b\} \ \subseteq \ \mathbb{R}^3$$

of the Example in the Introduction to this paper.

Let $f : I \to \mathbb{R}$ be a continuous function on an interval $I \subseteq \mathbb{R}$. Then f is semialgebraic (that is, its graph as a subset of $\mathbb{R}^2$ is semialgebraic) iff for some nonzero polynomial $P \in \mathbb{R}[x,y]$ we have $P\big(t, f(t)\big) = 0$ for all $t \in I$. (See [5, Chapter 2] for a detailed treatment of real semialgebraic sets and functions, in particular Exercise 3 in (3.7) there, with Hint on p. 169.) Thus if f is real analytic, and its restriction to some subinterval of I is semialgebraic, then f is semialgebraic.

Suppose f is semialgebraic and $I = (0, b)$ with $b \in (0, \infty]$. Then either $f(t) = 0$ for all sufficiently small $t > 0$, or for some $q \in \mathbb{Q}$ and $c \in \mathbb{R}^\times$ we have $f(t)/t^q \to c$ as $t \downarrow 0$. (See for example the end of [4].) Combining these facts we see that the following real analytic functions on $(0, \infty)$ cannot be semialgebraic on any subinterval of $(0, \infty)$: for $r \in \mathbb{R} \setminus \mathbb{Q}$ the function $t \mapsto t^r$; for $a \in (1, \infty)$ the function $t \mapsto a^t$; the function $t \mapsto \log t$.

By semialgebraic cell decomposition the algebraic part Y^{alg} of any set $Y \subseteq \mathbb{R}^n$ is the union of the sets $\mathrm{cl}(C) \cap Y$ with $C \subseteq Y$ a 1-dimensional semialgebraic cell. We can now prove the fact stated in the Introduction for the above $X \subseteq \mathbb{R}^3$ that

$$X^{\mathrm{alg}} \ := \bigcup_{q \in \mathbb{Q} \cap (1,2)} X_q.$$

The inclusion $\supseteq$ is clear. The sets X_q $(q \in \mathbb{Q} \cap (1,2))$ are closed in X, so given any 1-dimensional semialgebraic cell $C \subseteq X$ it suffices to show that $C \subseteq X_q$ for some $q \in \mathbb{Q} \cap (1,2)$. Now C is a $(0,0,1)$-cell, or a $(0,1,0)$-cell, or a $(1,0,0)$-cell. But X does not contain any $(0,0,1)$-cell. Also C cannot be a $(0,1,0)$-cell: if it were, then for a fixed $a \in (1,2)$ and an interval $I \subseteq (1,2)$ the function $t \mapsto a^t$ on I would be semialgebraic, which is false. Finally, suppose C is a $(1,0,0)$-cell. Then $C = \{\big(t, f(t), t^{f(t)}\big) : t \in I\}$ where $f : I \to \mathbb{R}$ is a continuous semialgebraic function on an interval $I \subseteq (1,2)$ (and $t \mapsto t^{f(t)} : I \to \mathbb{R}$ is also semialgebraic). Consider any interval $J \subseteq I$ on which f is of class C^1. Then taking the logarithmic derivative of $t \mapsto t^{f(t)} = \mathrm{e}^{f(t) \log t}$ on J gives that $f'(t) \log t + f(t)/t$ is semialgebraic as a function of $t \in J$, and so $f' = 0$ on J. Using several such J we see that f is constant on I. This constant value must be a rational $q \in (1,2)$, so $C \subseteq X_q$.

B. Some Model theory

In sections 6 and 7 we use the notion of an $\aleph_0$-saturated elementary extension, requiring a little excursion into model theory. We shall give precise definitions of the necessary model-theoretic notions, motivating them by examples, and stating carefully a few needed results. For most proofs we refer to [1, Appendix B]; there the basics of model theory are developed in the setting of many-sorted structures, while here we stay with one-sorted structures (which have only one underlying set, while a many-sorted structure has a *family* of underlying sets).

A *language* is a set L whose elements are called symbols, each symbol $s \in L$ being equipped with a natural number $\text{arity}(s) \in \mathbb{N}$. These symbols are either *relation symbols* or *function symbols*, and L is the disjoint union of L^{r}, its subset of relation symbols, and L^{f}, its subset of function symbols.

Below L is a language. Let $\mathcal{M}$ be an *L-structure*, that is, a triple

$$\mathcal{M} \;=\; \big(M; (R^{\mathcal{M}})_{R\in L^{\mathrm{r}}}, (F^{\mathcal{M}})_{F\in L^{\mathrm{f}}}\big)$$

consisting of a *nonempty* set M, an m-ary relation $R^{\mathcal{M}} \subseteq M^m$ on M for each $R \in L^{\mathrm{r}}$ of arity m, and an n-ary function $F^{\mathcal{M}} : M^n \to M$ on M for each $F \in L^{\mathrm{f}}$ of arity n. We call M the *underlying set* of $\mathcal{M}$, we think of a symbol $R \in L^{\mathrm{r}}$ as *naming* the corresponding relation $R^{\mathcal{M}}$ on M, and likewise for $F \in L^{\mathrm{f}}$.

Thus a nullary function symbol (also called a *constant symbol*) names a function $M^0 \to M$, to be identified with its unique value in M, so a constant symbol names a distinguished element of M. Usually we drop the superscripts $\mathcal{M}$ in $R^{\mathcal{M}}$ and $F^{\mathcal{M}}$ when $\mathcal{M}$ is understood from the context, the distinction between the symbols and what they name to be kept in mind. We shall also feel free to denote $\mathcal{M}$ and its underlying set M by the same letter, when convenient. The reason we need the distinction between symbols and what they name in a particular L-structure is that we have to be able to say that a statement expressed in terms of these symbols holds in, say, two *different* L-structures $\mathcal{M}$ and $\mathcal{N}$.

We need to consider two (unrelated) ways of increasing $\mathcal{M}$. The *first* is when L is a sublanguage of L'. Then an *L'-expansion of* $\mathcal{M}$ is an L'-structure $\mathcal{M}'$ with the same underlying set as $\mathcal{M}$ and where the symbols of L name the same relations and functions in $\mathcal{M}$ as in $\mathcal{M}'$. We also say that then $\mathcal{M}'$ *expands* $\mathcal{M}$.

Example. The language L_{OF} of ordered fields has a binary relation symbol $<$, constant symbols 0 and 1, a unary function symbol $-$, and binary function symbols $+$ and $\cdot$. Any ordered field K is viewed as an L_{OF}-structure by having $<$ name the (strict) ordering of the field, and the function symbols name the functions on K that these symbols traditionally denote. Thus an ordered field K *expands* its underlying field. Equipping the ordered field $\mathbb{R}$ of real numbers with the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ gives the expansion $\mathbb{R}_{\exp}$ of $\mathbb{R}$. This is not exactly how we specified $\mathbb{R}_{\exp}$ in part A of the appendix, but the difference is immaterial: the two descriptions lead to the same sets $X \subseteq \mathbb{R}^n$ being definable in $\mathbb{R}_{\exp}$, see below. For *model-theoretic* use we take $\mathbb{R}_{\exp}$ as the above expansion of $\mathbb{R}$.

A *second* way: $\mathcal{M}$ *is a substructure of* $\mathcal{N}$ (or $\mathcal{N}$ *is an extension of* $\mathcal{M}$); notation: $\mathcal{M} \subseteq \mathcal{N}$. This means: $\mathcal{N} = (N; \dots)$ is an L-structure, $M \subseteq N$, $R^{\mathcal{N}} \cap M^m = R^{\mathcal{M}}$ for m-ary $R \in L^{\mathrm{r}}$, and $F^{\mathcal{M}}(a) = F^{\mathcal{N}}(a)$ for n-ary $F \in L^{\mathrm{f}}$ and $a \in M^n$. For example, if K_1 and K_2 are ordered fields viewed as L_{OF}-structures, $K_1 \subseteq K_2$ means that K_1 is an ordered subfield of K_2 (so the ordering of K_2 restricts to the ordering of K_1).

The 0-definable (or absolutely definable) sets of $\mathcal{M}$ are the sets $X \subseteq M^n$ for $n = 0, 1, 2, \ldots$ obtained recursively as follows:

(D1) $R^{\mathcal{M}} \subseteq M^m$ for m-ary $R \in L^{\mathrm{r}}$ and $\mathrm{graph}(F^{\mathcal{M}}) \subseteq M^{n+1}$ for n-ary $F \in L^{\mathrm{f}}$ are 0-definable.
(D2) if $X, Y \subseteq M^n$ are 0-definable, then so are $X \cup Y$ and $M^n \setminus X$;
(D3) if $X \subseteq M^n$ is 0-definable, then so are $X \times M, M \times X \subseteq M^{n+1}$;
(D4) for any $i < j$ in $\{1, \ldots, n\}$ the diagonal $\{(a_1, \ldots, a_n) : \ a_i = a_j\} \subseteq M^n$ is 0-definable;
(D5) if $X \subseteq M^{n+1}$ is 0-definable, then so is $\pi(X) \subseteq M^n$ where $\pi : M^{n+1} \to M^n$ is given by $\pi(a_1, \ldots, a_n, a_{n+1}) = (a_1, \ldots, a_n)$.

Thus the 0-definable sets $X \subseteq M^n$ are exactly those that belong to the smallest structure on M that contains all sets $R^{\mathcal{M}}$ with $R \in L^{\mathrm{r}}$ and all sets $\mathrm{graph}(F^{\mathcal{M}})$ with $F \in L^{\mathrm{f}}$. If $X \subseteq M^n$ is a 0-definable set of $\mathcal{M}$ and the ambient $\mathcal{M}$ is clear from the context, we also just say that X is 0-definable. A map $f : X \to M^n$ with $X \subseteq M^m$ is said to be 0-definable (in $\mathcal{M}$) if its graph as a subset of M^{m+n} is 0-definable in $\mathcal{M}$. In that case its domain X is 0-definable and for every 0-definable $X' \subseteq X$ its image $f(X') \subseteq M^n$ is 0-definable.

We need a notation system to describe 0-definable sets in a uniform way in varying L-structures. Towards this we assume that in addition to the symbols of L we have an infinite set Var of symbols, called *variables* (with Var disjoint from the language L and independent of L). Given a tuple $x = (x_1, \ldots x_m)$ of distinct variables we define L-terms t in x recursively as follows: each x_i with $1 \leqslant i \leqslant m$ is an L-term in x, and for n-ary $F \in L^{\mathrm{f}}$ and L-terms $t_1, \ldots, t_n$ in x, the expression $F(t_1, \ldots, t_n)$ is an L-term in x. Formally, these expressions are words on some alphabet together with the specification of the tuple x, but we prefer not to go into detail on such syntactical matters. We let $t(x)$ indicate a term t in x. An L-term $t = t(x)$ names in an obvious way a function $t^{\mathcal{M}} : M^m \to M$, with x_i naming the function $(a_1, \ldots, a_m) \mapsto a_i : M^m \to M$. Functions named by L-terms are 0-definable in $\mathcal{M}$.

For example, when K is an ordered field, any L-term t in $x = (x_1, \ldots, x_m)$ names a function $K^m \to K$ given by a polynomial in $\mathbb{Z}[x_1, \ldots, x_m]$, and each such polynomial function is named by an L-term in x. (Different terms can name the same function: $x + (-x)$ and 0 are different as terms in the single variable x, but name the same function $K \to K$, which takes the constant value 0.)

Going back to our L-structure $\mathcal{M}$ we introduce for every element $a \in M$ a constant symbol $\underline{a} \notin L$ as a name for a, with $\underline{a} \neq \underline{b}$ for all $a \neq b$ in M. For every set $A \subseteq M$ we extend L to the language $L_A := L \cup \{\underline{a} : \ a \in A\}$, and expand $\mathcal{M}$ to the L_A-structure $\mathcal{M}_A$ with $\underline{a}$ naming a for $a \in A$. A set $X \subseteq M^n$ is said to be A-definable (or definable over A) in $\mathcal{M}$ if X is 0-definable in $\mathcal{M}_A$. When $A = M$ we just use write "definable" instead of "M-definable". A set $X \subseteq M^n$ is A-definable (in $\mathcal{M}$) iff $X = Y(a)$ for some 0-definable set $Y \subseteq M^{m+n}$ and some $a \in A^m$. (Here for $Y \subseteq M^{m+n}$ and $a \in M^m$ we set $Y(a) := \{b \in M^n : \ (a, b) \in Y\}$.)

Examples. For an algebraically closed field $\boldsymbol{k}$, viewed as an L-structure with $L = \{0, 1, -, +, \cdot\}$ (the language of rings), the subsets of $\boldsymbol{k}^n$ definable in $\boldsymbol{k}$ are exactly the so-called constructible subsets of $\boldsymbol{k}^n$: the finite unions of sets $X \setminus Y$ with X, Y Zariski-closed subsets of $\boldsymbol{k}^n$. (This is basically the constructibility theorem of Tarski-Chevalley: the image of a constructible subset of $\boldsymbol{k}^{n+1}$ in $\boldsymbol{k}^n$ under the projection map $(a_1, \ldots, a_{n+1}) \mapsto (a_1, \ldots, a_n) : \boldsymbol{k}^{n+1} \to \boldsymbol{k}^n$ is constructible in $\boldsymbol{k}^n$.)

The same holds with "A-definable" and "A-constructible" instead of "definable" and "constructible" for any set $A \subseteq \boldsymbol{k}$, where an A-constructible subset of $\boldsymbol{k}^n$ is a finite union of sets $X \setminus Y$ with $X, Y \subseteq \boldsymbol{k}^n$ given by the vanishing of polynomials in $D[x_1, \dots, x_n]$ where D is the subring of $\boldsymbol{k}$ generated by A.

For us the more relevant example is when K is an ordered real closed field. Then the subsets of K^n definable in K are exactly the semialgebraic subsets of K^n, that is, the finite unions of sets (with $f, g_1, \dots, g_m$ ranging over $K[x_1, \dots, x_n]$)

$$\{a \in K^n : \ f(a) = 0,\ g_1(a) > 0, \dots, g_m(a) > 0\}.$$

This is the Tarski-Seidenberg theorem (like the Tarski-Chevalley theorem, but with "semialgebraic" instead of "constructible"). Requiring the polynomials $f, g_1, \dots, g_m$ above to have coefficients in $\mathbb{Z}$, we obtain likewise exactly the subsets of K^n that are 0-definable in K.

Saturation. This notion functions as a kind of compactness for definable sets. Let κ be a cardinal. An L-structure $\mathcal{M}$ is said to be *κ-saturated* if for every set $A \subseteq M$ of cardinality $< \kappa$ and any family (X_i) of A-definable subsets of M with the finite intersection property we have $\bigcap_i X_i \neq \emptyset$. (The *finite intersection property for* (X_i) says that $X_{i_1} \cap \dots \cap X_{i_n} \neq \emptyset$ for all indices $i_1, \dots, i_n$.) This property of families of definable subsets of M is inherited by families of definable subsets of M^m for any m. One can indeed think of this in terms of compactness: if $\mathcal{M}$ is κ-saturated, then for $A \subseteq M$ of cardinality $< \kappa$, the A-definable subsets of M^m are a basis for a topology on M^m, the A-topology, which makes M^m a compact hausdorff space in which these A-definable sets are exactly the open-and-closed sets. We need this only for $\kappa = \aleph_0$, which means that in the definition above we restrict to *finite* $A \subseteq M$. For $\kappa = \aleph_1$ the restriction is to *countable* $A \subseteq M$.

For example, any algebraically closed field of infinite transcendence degree over its prime field is $\aleph_0$-saturated, and the field $\mathbb{C}$ of complex numbers is even $\aleph_1$-saturated. The ordered field $\mathbb{R}$ is not even 1-saturated, since $\bigcap_n (n, \infty) = \emptyset$. Any finite structure (a structure with finite underlying set) is κ-saturated for every κ.

Towards the study of a structure $\mathcal{M}$ of interest we can always pass to an $\aleph_1$-saturated extension $\mathcal{N}$ with the same elementary properties, do our work in $\mathcal{N}$ and then pass the information gained back to $\mathcal{M}$. This will be made precise below. To make sense of "the same elementary properties" we need a notation system for definable sets. This is the reason for introducing formulas and sentences below.

Formulas and sentences. Let $y = (y_1, \dots, y_n)$ be a tuple of distinct variables. We define L-formulas ϕ in y inductively as follows:

(i) The *atomic* L-formulas in y are the expressions

$$\top, \quad \perp, \quad R\big(t_1, \dots, t_m\big), \quad \text{and } t_1 = t_2$$

for m-ary $R \in L^{\mathrm{r}}$ and L-terms $t_1, \dots, t_m$ in y, and L-terms t_1, t_2 in y.

(ii) Given any L-formulas ϕ and ψ in y, we have new L-formulas in y:

$$\neg\phi, \quad \phi \vee \psi, \quad \text{and } \phi \wedge \psi.$$

(iii) Let ϕ be a formula in $(y_1, \dots, y_i, z, y_{i+1}, \dots, y_n)$, where $0 \leqslant i \leqslant n$ and z is a variable different from $y_1, \dots, y_n$; then

$$\exists z\phi \ \text{ and } \ \forall z\phi$$

are new L-formulas in y.

Formally, these formulas in y are words on a certain alphabet, together with the specification of the tuple y. We also write $\phi(y)$ to indicate that we are dealing with a formula ϕ in y. Each L-formula $\phi = \phi(y)$ names (we also say: *defines*) a 0-definable set $\phi^{\mathcal{M}} \subseteq M^n$: the atomic formulas $\top$ and $\perp$ name the subsets M^n and $\emptyset$ of M^n, and the atomic formulas $R(t_1, \dots, t_m)$ and $t_1 = t_2$ as above name the sets

$$\{a \in M^n : (t_1^{\mathcal{M}}(a), \dots, t_m^{\mathcal{M}}(a)) \in R^{\mathcal{M}}\} \text{ and } \{a \in M^n : t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a)\},$$

and for L-formulas ϕ, ψ as above,

$$(\neg\phi)^{\mathcal{M}} = M^n \setminus \phi^{\mathcal{M}}, \quad (\phi \vee \psi)^{\mathcal{M}} := \phi^{\mathcal{M}} \cup \psi^{\mathcal{M}}, \ (\phi \wedge \psi)^{\mathcal{M}} = \phi^{\mathcal{M}} \cap \psi^{\mathcal{M}},$$

and for an L-formula ϕ in $(y_1, \dots, y_i, z, y_{i+1}, \dots, y_n)$ as above: $(\exists z \phi)^{\mathcal{M}}$ is the set

$$\{(a_1, \dots, a_n) \in M^n : (a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \in \phi^{\mathcal{M}} \text{ for some } b \in M\},$$

that is, the image of $\phi^{\mathcal{M}} \subseteq M^{n+1}$ under the projection map

$$(a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \mapsto (a_1, \dots, a_n) : M^{n+1} \to M^n,$$

and $(\forall z \phi)^{\mathcal{M}} := (\neg\exists z \neg\phi)^{\mathcal{M}}$, which equals the set

$$\{(a_1, \dots, a_n) \in M^n : (a_1, \dots, a_i, b, a_{i+1}, \dots, a_n) \in \phi^{\mathcal{M}} \text{ for every } b \in M\}.$$

The 0-definable subsets of M^n are exactly the sets $\phi^{\mathcal{M}}$ with ϕ an L-formula in y. Likewise for $A \subseteq M$, the A-definable subsets of M^n are exactly the sets $\phi^{\mathcal{M}_A}$ with ϕ an L_A-formula in y, but for convenience we write this also as $\phi^{\mathcal{M}}$.

The L-formulas ϕ in $y = (y_1, \dots, y_n)$ with $n = 0$ have a special status and are called *L-sentences*, typically denoted by σ. One can think of a sentence as making an assertion. Formally, a sentence σ names a set $\sigma^{\mathcal{M}} \subseteq M^0$, so it equals M^0, in which case we say that σ is *true in* $\mathcal{M}$ (notation: $\mathcal{M} \models \sigma$), or it is empty, in which case we way that σ is *false in* $\mathcal{M}$. For example, if L is the language of rings and x, y are distinct variables, then the L-sentence $\forall x \exists y (x = y \cdot y)$ is true in exactly those fields in which every element is a square.

Elementary extensions. An *elementary extension* of the L-structure $\mathcal{M}$ is an extension $\mathcal{N} \supseteq \mathcal{M}$ of $\mathcal{M}$ such that the same L_M-sentences are true in $\mathcal{M}$ and $\mathcal{N}$ (where of course for $a \in M$ the constant symbol $\underline{a}$ names a in both $\mathcal{M}$ and $\mathcal{N}$). Notation: $\mathcal{M} \preccurlyeq \mathcal{N}$. Here are two wellknown situations where this is the case: any algebraically closed field is an elementary extension of any algebraically closed subfield, any real closed field is an elementary extension of any real closed subfield.

Suppose $\mathcal{M} \preccurlyeq \mathcal{N}$. Then for any L_M-formula $\phi = \phi(x_1, \dots, x_n)$ we have $\phi^{\mathcal{M}} = \phi^{\mathcal{N}} \cap M^n$. Moreover, if $\psi = \psi(x_1, \dots, x_n)$ is a second L_M-formula and $\phi^{\mathcal{M}} = \psi^{\mathcal{M}}$, then $\phi^{\mathcal{N}} = \psi^{\mathcal{N}}$, so a definable set $X = \phi^{\mathcal{M}} \subseteq M^n$ (of $\mathcal{M}$) yields a definable set $X(\mathcal{N}) = \phi^{\mathcal{N}} \subseteq N^n$ that does not depend on the choice of defining formula ϕ.

To profit from saturation and elementary extensions we use:

Proposition B.1. *Any L-structure has an $\aleph_1$-saturated elementary extension.*

Indeed, for any L-structure $\mathcal{M}$ and nonprincipal ultrafilter U on the set $\mathbb{N}$, the ultrapower $\mathcal{M}^{\mathbb{N}}/U$ is an $\aleph_1$-saturated elementary extension of $\mathcal{M}$, where $\mathcal{M}$ is identified with a substructure of $\mathcal{M}^{\mathbb{N}}/U$ via the diagonal embedding; this is a remark intended for those who know about ultrapowers. In Sections 6 and 7 of this paper we only need $\aleph_0$-saturation instead of the stronger $\aleph_1$-saturation.

Revisiting o-minimality. Let K be an *expansion* of an ordered field. Among the definable subsets of K in this expansion are at least the open intervals $(a,b)_K$ with $a<b$ in $K\cup\{-\infty,+\infty\}$, and thus the finite unions of such open intervals and singletons $\{a\}$ with $a\in K$. We say that K is *o-minimal* if there are no other subsets of K definable in this expansion. To see how this is related to the concept of o-minimality considered in the first part of this appendix, let $\mathrm{Def}^n(K)$ be the collection of sets $X\subseteq K^n$ that are definable in this expansion K. Note that then K is o-minimal if and only if the family $\big(\mathrm{Def}^n(K)\big)$ is an o-minimal structure on the underlying ordered field of K. In particular, if K is o-minimal, then the underlying ordered field of K is real closed.

Lemma B.2. *If K is o-minimal, then so is any elementary extension of K.*

Proof. Assume K is o-minimal, and $K\preccurlyeq K^*$, so the underlying ordered field of K^* extends the underlying ordered field of K. Let $X\subseteq K^*$ be definable. Then we have a definable set $Y\subseteq K^{n+1}$ and a point $b\in (K^*)^n$ such that $X=Y^{K^*}(b)$. Take $N\in\mathbb{N}$ and cells $C_1,\dots,C_N$ in K^{n+1} such that $Y=C_1\cup\cdots\cup C_N$. Then for every $a\in K^n$ the set $Y(a)$ is a union of at most N sets $\{c\}$ with $c\in K$, and intervals of K. With K an L-structure, this fact can be expressed by a certain L_K-sentence being true in K, hence in K^*, which then means in particular that $X=Y^{K^*}(b)$ is a union of at most N sets $\{c\}$ with $c\in K^*$, and intervals of K^*. □

Let K be o-minimal and $K\preccurlyeq K^*$. To each definable set $X\subseteq K^n$ we associate the set $X^*:=X(K^*)\subseteq (K^*)^n$, which is definable (over K) in K^*. If $C\subseteq K^n$ is an $(i_1,\dots,i_n)$-cell in the sense of K, then $C^*\subseteq (K^*)^n$ is an $(i_1,\dots,i_n)$-cell in the sense of K^*. It follows that for definable $X\subseteq K^n$ we have $\dim X=\dim X^*$ where the dimension on the left is in the sense of K, and the dimension on the right is in the sense of K^*.

In the main text and in part A of this appendix we used the letter R to refer to an o-minimal field, but in this part B we used R to indicate a relation symbol. That is why in this part B we prefer the letter K when dealing with o-minimal expansions of ordered fields and o-minimal fields. The distinction between the two concepts (*o-minimal expansion of an ordered field* and *o-minimal field*) is often immaterial: we saw that an o-minimal expansion K of an ordered field gives rise to an o-minimal field with the same underlying ordered field and the same definable sets.

When considering *elementary extensions* and $\aleph_0$-*saturation*, the distinction is significant: When referring to an *elementary extension of an o-minimal field* we really mean an elementary extension of an o-minimal expansion K of an ordered field, so K is then an L-structure for a suitable language $L\supseteq L_{\mathrm{OF}}$. Likewise when referring to an o-minimal field as being $\aleph_0$-saturated, we mean: an L-structure K that gives rise to this o-minimal field is $\aleph_0$-saturated.

How to use the above? To illustrate typical uses, we first show how Theorem 4.1 follows from it being true when the ambient o-minimal field is $\aleph_0$-saturated. So let K be an o-minimal field, $X\subseteq K^m$ a strongly bounded definable set, and $k\geqslant 1$; we need to show that X has a k-parametrization. We can assume $X\neq\emptyset$, and set $l:=\dim X$. Take $N\in\mathbb{N}$ such that $X\subseteq [-N,N]_K^m$. As explained, K is viewed as an L-structure for a language $L\supseteq L_{\mathrm{OF}}$. Fix an L_K-formula $\phi(x)$, $x=(x_1,\dots,x_m)$, such that $X=\phi^K$. Take an $\aleph_0$-saturated elementary extension K^* of K. Then K^*

is an o-minimal field and

$$X^* = \phi^{K^*} \subseteq [-N,N]^m_{K^*}$$

is strongly bounded, and so has a k-parametrization $\{f_1,\dots,f_M\}$ (with respect to K^*), since Theorem 4.1 was established in Section 7 when the ambient o-minimal field is $\aleph_0$-saturated. For $\mu = 1,\dots,M$ we have $f_\mu : (0,1)^l_{K^*} \to (K^*)^m$, so the graph of f_μ is defined in K^* by an L_{K^*}-formula $\phi_\mu(b,v,x)$ with $\phi_\mu = \phi_\mu(u,v,x)$ an L_K-formula, $u = (u_1,\dots,u_p)$, $v = (v_1,\dots,v_l)$ and $b \in (K^*)^p$ (the same p and b for all μ, without loss of generality). The fact that there exists $b \in (K^*)^p$ such that $\phi_1(b,v,x),\dots,\phi_M(b,v,x)$ define in K^* the graphs of functions of a k-parametrization of X^* can be expressed by a certain L_K-sentence $\exists u\theta(u)$ being true in K^*. (This sentence is complicated but its construction is routine and just mimicks the definitions of the various notions involved.) Hence this sentence $\exists u\theta(u)$ is also true in K, which then means that for some $a \in K^m$ the formulas $\phi_1(a,v,x),\dots,\phi_M(a,v,x)$ define in K the graphs of functions of a k-parametrization of X.

In a very similar way Theorem 4.2 follows from the fact, established in Section 7, that it is true when the ambient o-minimal field is $\aleph_0$-saturated.

Next we explain the use of "$\aleph_0$-saturation plus Definable Selection" in obtaining Corollary 7.2 as a consequence of Corollary 7.1. (Other uses of this device, in the proof of Lemma 6.2 and earlier in Section 7 are along the same lines. The argument we give may seem lengthy, but such arguments are utterly routine in model theory and are therefore usually not spelled out but left to the reader.)

We are now dealing with an o-minimal field K which is $\aleph_0$-saturated when viewed as an L-structure for suitable $L \supseteq L_{\mathrm{OF}}$ as before. We are given d,k,m,n with $k,n \geqslant 1$ and definable $E \subseteq K^m$ and definable $Z \subseteq E \times [-1,1]^n \subseteq K^{m+n}$ with $\dim Z(s) = d$ for all $s \in E$. Take a finite $A \subseteq K$ such that E and Z are A-definable. Corollary 7.1 yields for every $s \in E$ a definable C^k-map

$$f = (f_1,\dots,f_N) : (0,1)^d \to (K^n)^N = K^{Nn}$$

with $N = N(s) \in \mathbb{N}$ depending on s, such that (i) and (ii) of that corollary hold for $\Phi := \{f_1,\dots,f_N\}$ and $X(s)$ in the role of X. For each L-formula $\phi = \phi(u,x,y)$,

$$u = (u_1,\dots,u_M), \quad x = (x_1,\dots,x_d), \quad y = (y_{11},\dots,y_{Nn})$$

with $M,N \in \mathbb{N}$ depending on ϕ we consider the A-definable set $E_\phi \subseteq K^m$ of all $s \in E$ such that for some $b \in K^M$ the L_K-formula $\phi(b,x,y)$ defines the graph of a map f parametrizing $X(s)$ as above. Since K is $\aleph_0$-saturated and E is covered by the sets E_ϕ, it is covered by finitely many of them, say by $E_{\phi_1},\dots,E_{\phi_e}$,

$$\phi_i = \phi_i(u_1,\dots,u_{M(i)},x,y_{11},\dots,y_{N(i)n}) \qquad (i=1,\dots,e).$$

For $i=1,\dots,e$, let $E(i)$ be the definable set of all $s \in E$ with $s \in E_{\phi_i}$ and $s \notin E_{\phi_j}$ for $1 \leqslant j < i$. Definable selection then yields for such i a definable map

$$s \mapsto b_i(s) : E(i) \to K^{M(i)}$$

such that for every $s \in E(i)$ the L_K-formula $\phi_i(b_i(s),x,y)$ defines in K the graph of a C^k-map $f : (0,1)^d \to (K^n)^{N(i)}$ parametrizing $X(s)$ as specified earlier. Now E is the disjoint union of $E(1),\dots,E(e)$. Adding suitable constant maps it is routine to obtain from this for $N = \max_i N(i)$ a definable set $F \subseteq E \times K^d \times K^{Nn}$ for which the conclusion of Corollary 7.2 holds.

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